Calculus 1 Workbook and Mathematica Applets

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Georgia State University, Dunwoody Campus

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Introduction

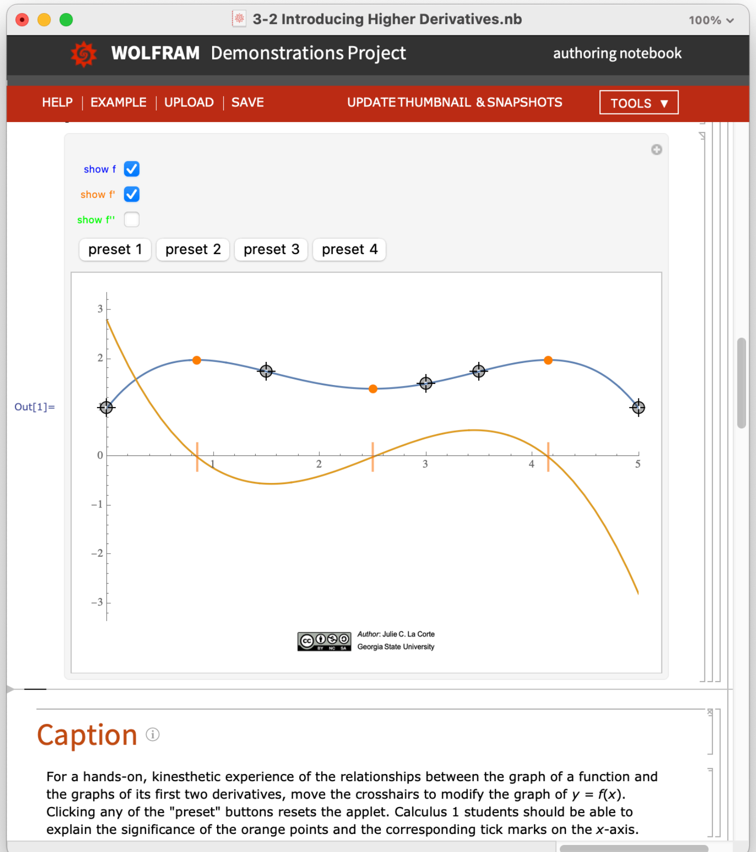
Today I'll be presenting work completed for a grant to produce auxiliary materials for use with open source textbooks.

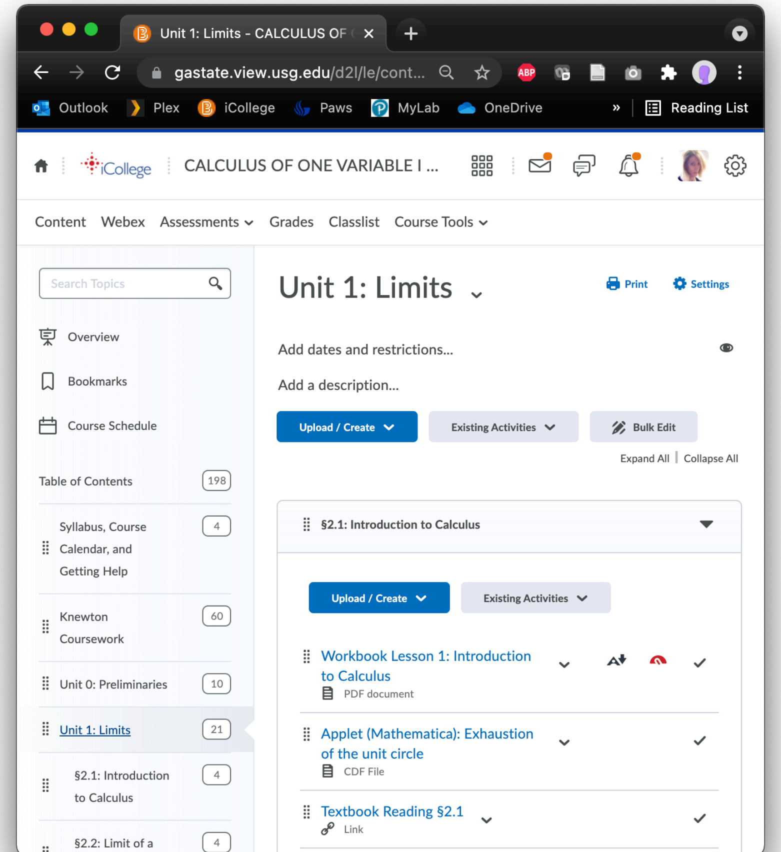
- The grant was provided by Affordable Learning Georgia.
- Invaluable assistance with the grant process was provided by Leonard and Glenn at Perimeter College's Office of Grants Development and Administration.

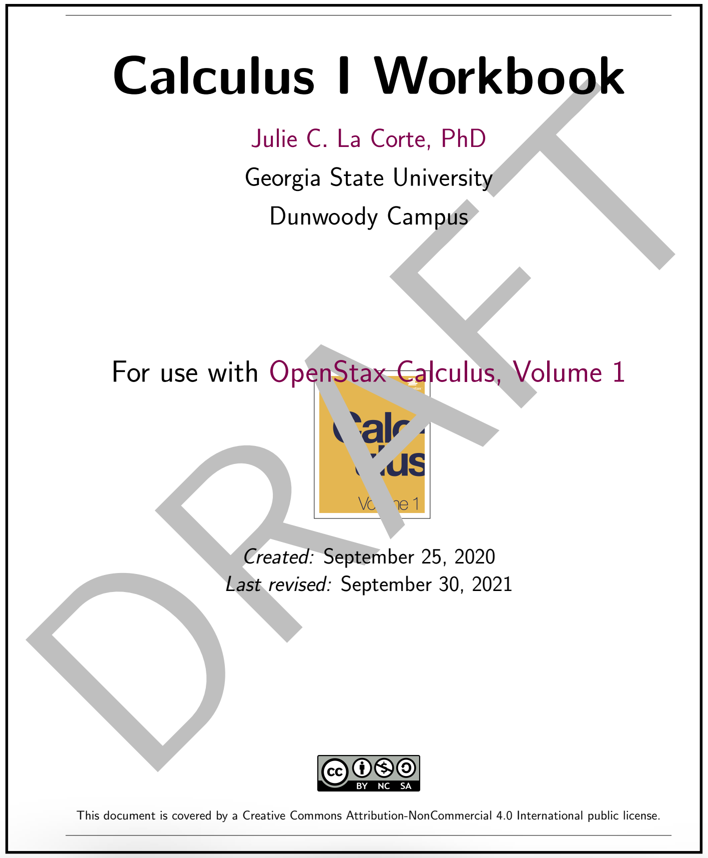
I produced Mathematica applets, a Brightspace/iCollege module, and a Workbook (PDF) for **Calculus 1**.

- The materials were first piloted in Spring 2021 (enrollment: 5).
- I continue to teach with and revise them.

Materials produced







I produced Mathematica applets, a Brightspace/iCollege Module, and a Workbook (PDF) for **Calculus 1**.

- The materials were first piloted in Spring 2021 (enrollment: 5).
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Project narrative

Motivation for the project

"Improving Student Performance in Calculus"
Speakers: James Williams, Behnaz Rouhani, Somaya Muiny
Perimeter College Faculty Development Day, Oct. 18, 2019

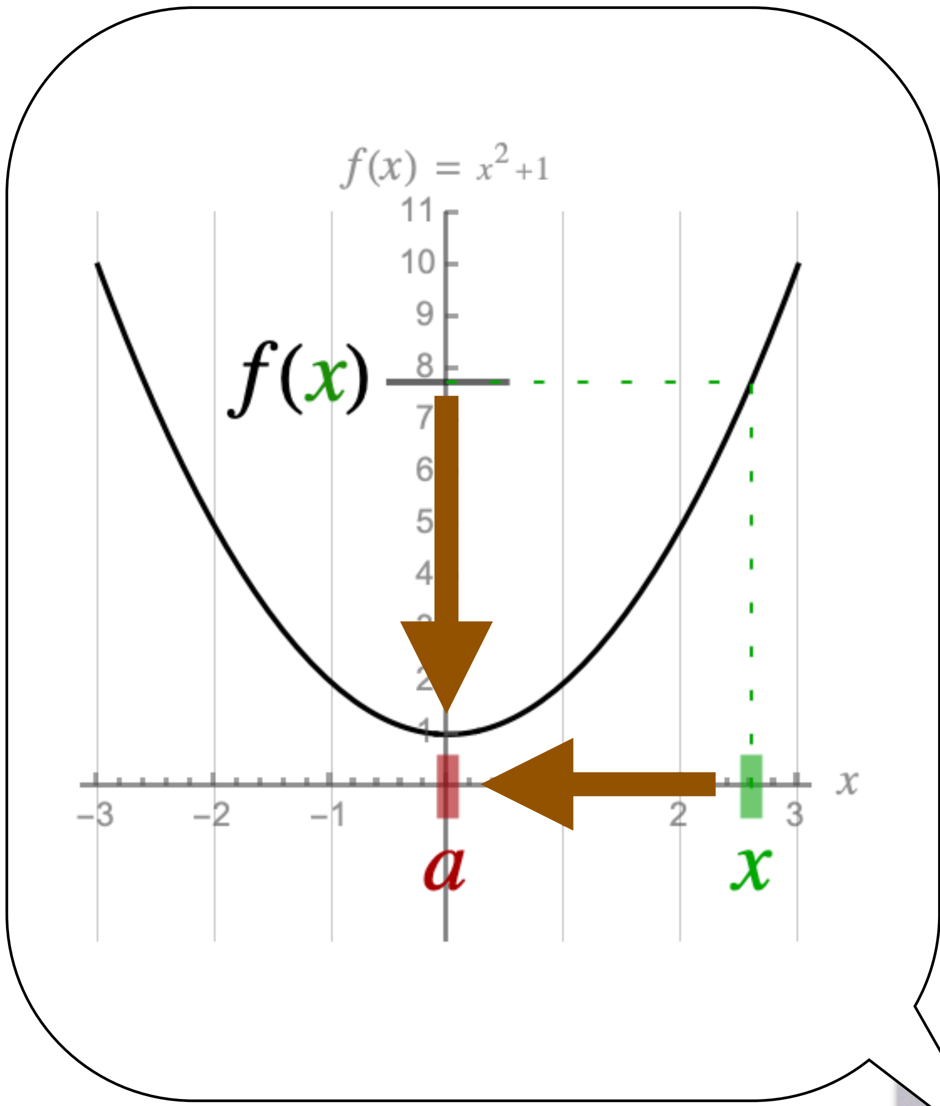
Multiple choice assessments were given in Spring 2018 and Spring 2019

Problem areas in Calculus 1:

- Use the graph of $y = f(x)$ to determine the lefthand limit as $x \to 1^-$
  —In 2018, 75% answered correctly
  —In 2019, 65% answered correctly
- Optimization problem: Find dimensions that maximize the area of a rectangular region.
  —In 2018, 55.1% answered correctly
  —In 2019, 60.8% answered correctly
  —In 2018, statement of problem was "very wordy"
  —"Performance went up when we added a picture."

Project narrative

Skills for exercises in problem areas



Part of my job as a teacher is to get my students to see the pictures I have in my head.

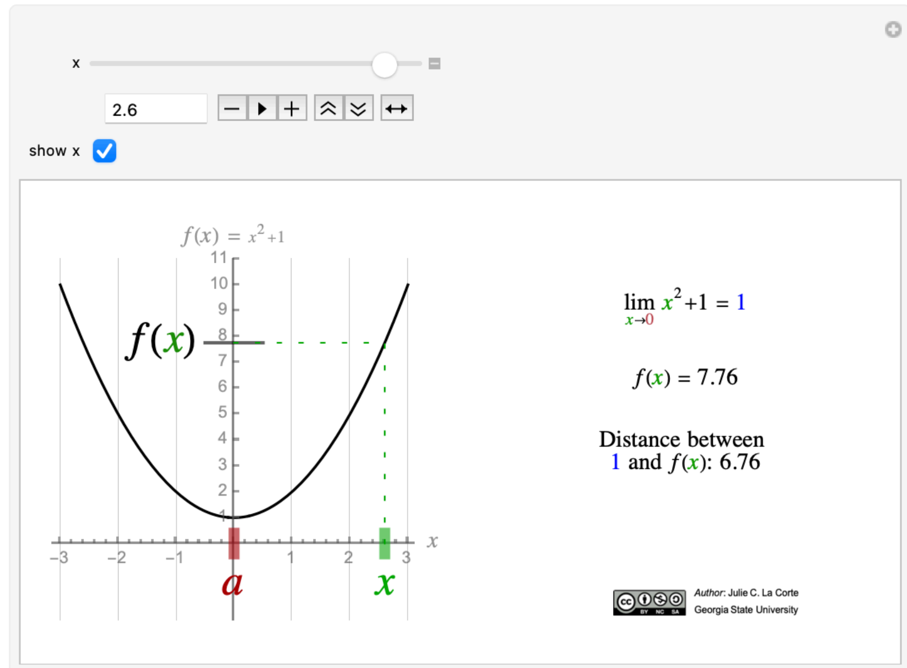
These pictures often have moving parts.

Project narrative

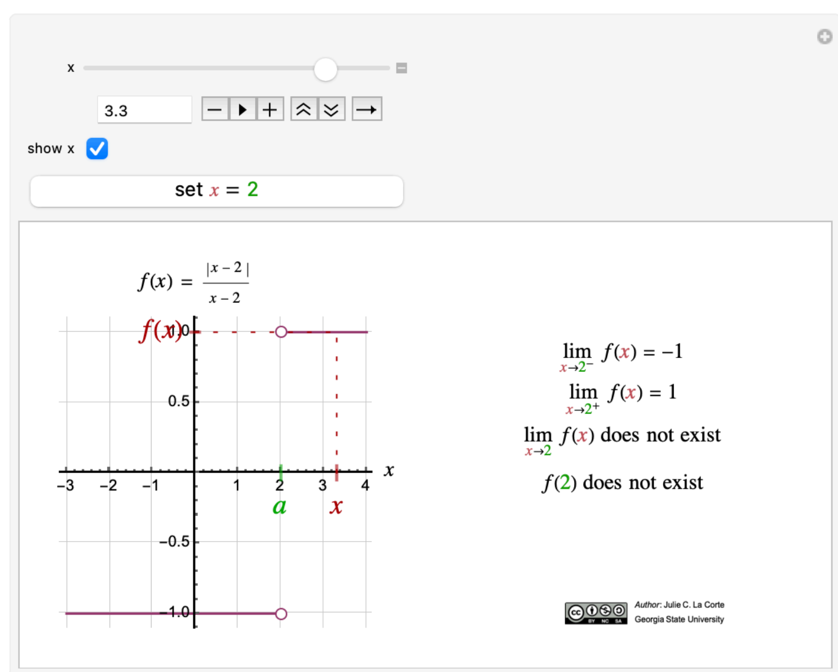
Skills for exercises in problem areas

Finding the limit of a function from its graph:

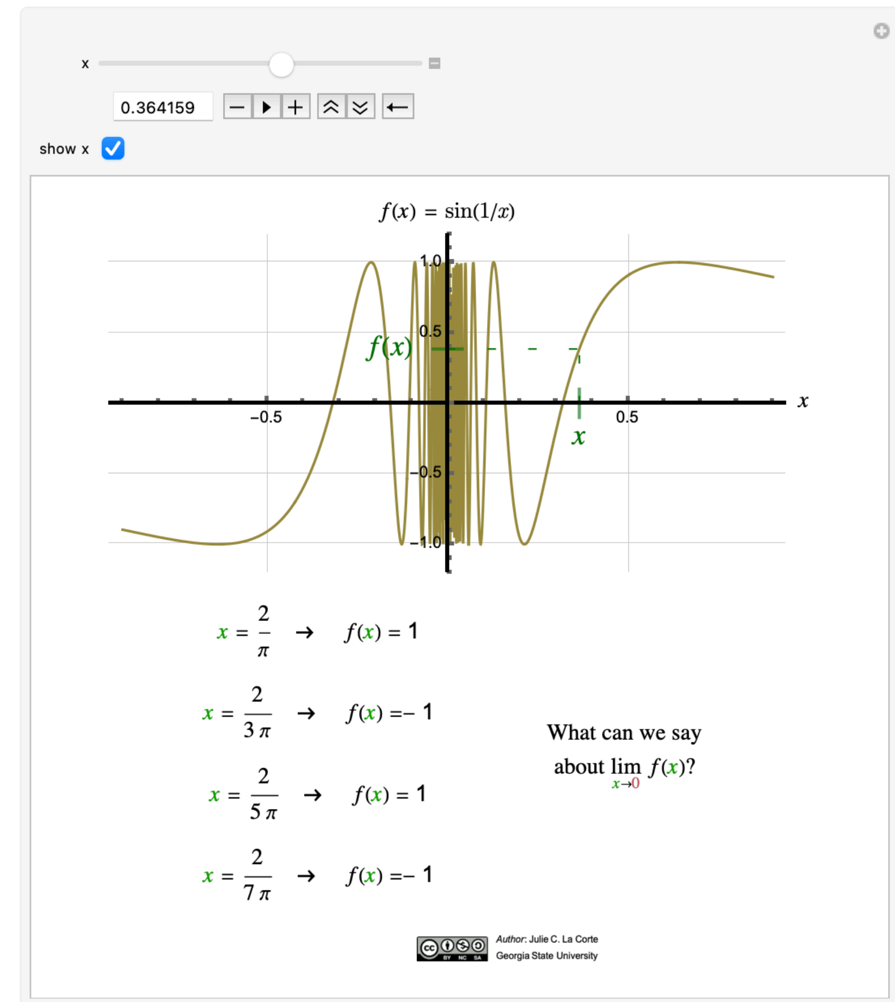
- graphical
- analytic
- kinesthetic



§2.2, "First example of a limit"



§2.2, "Visualizing one-sided and two-sided limits"



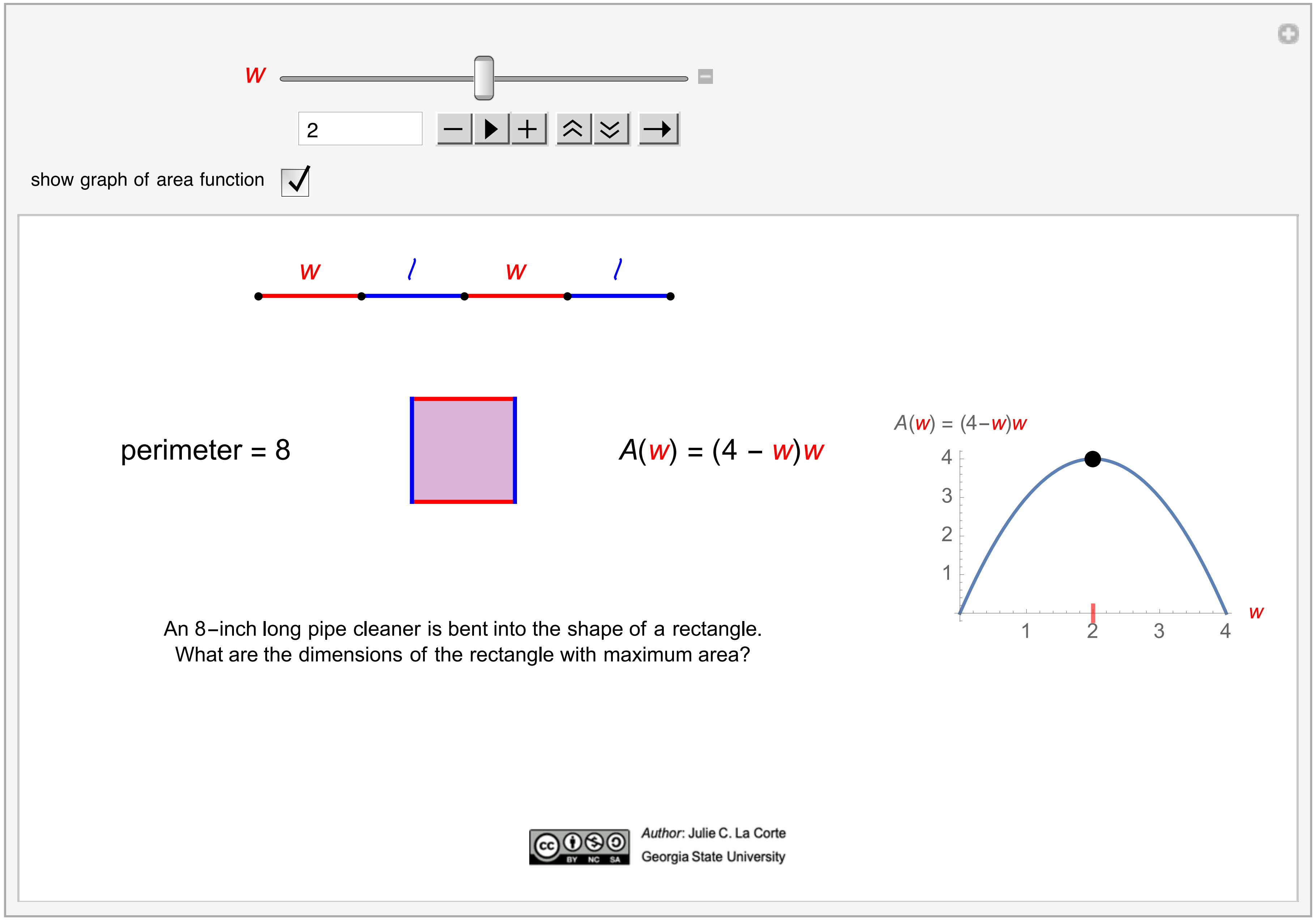
§2.5, "A limit that does not exist"

Project narrative

Skills for exercises in problem areas

Optimization problems:

- strategic
- analytic
- graphical
- kinesthetic



§4.7, "Applied optimization problems"

Project narrative

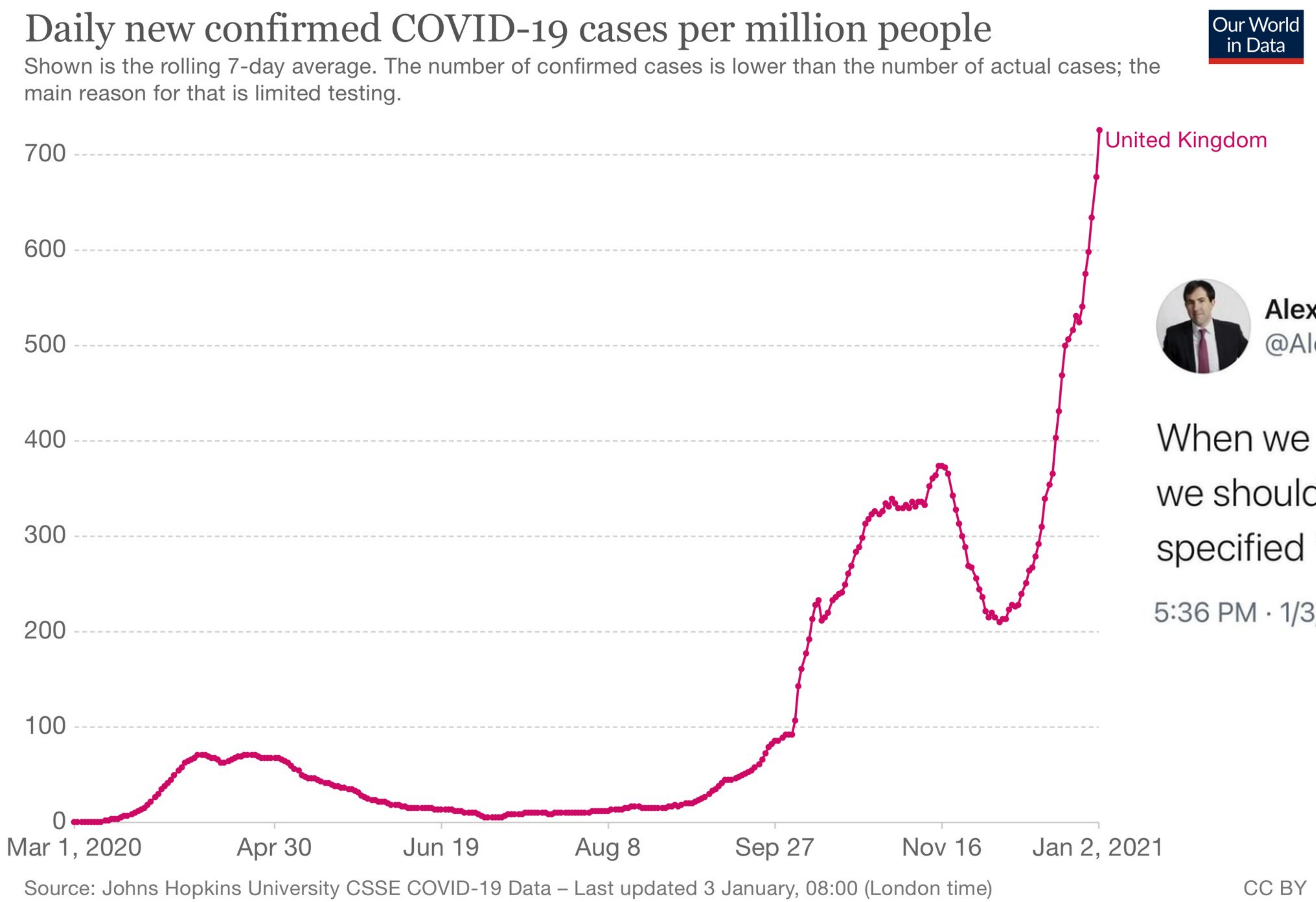
Original plan

Original plan:

- Create a few applets that provide hands-on experience with challenging concepts and techniques
- **Animation:** Show the "moving parts" of definitions and exercises *actually moving*
- **Manipulation:** Let students interact via sliders, click to add to a picture, drag to change a picture, start and stop animations, ...
- **Abstraction:** Facilitate "big picture" insights through pictures, animations, and discovery through experimentation

Project narrative

Pandemic



Project narrative

Original plan

New plan!

- Provide a self-contained course in the form of a **Workbook** suitable for use with various or mixed methods (face-to-face, "hybrid," online)
- Integrate the applets into the text of the Workbook
- Build a **Brightspace/iCollege Course Module** where all course materials can be accessed
- **"Sandbox" design:** facilitate unguided exploration; make applets as inviting and self-explanatory as possible

Project narrative

Timeline

semesters of grant
Summer 2020
Fall 2020
Spring 2021

Grant funding period

Classroom pilot

semesters of pilot	enrollment
Spring 2021	5
Summer 2021	11
Fall 2021	14

Course materials

Applets and Workbook

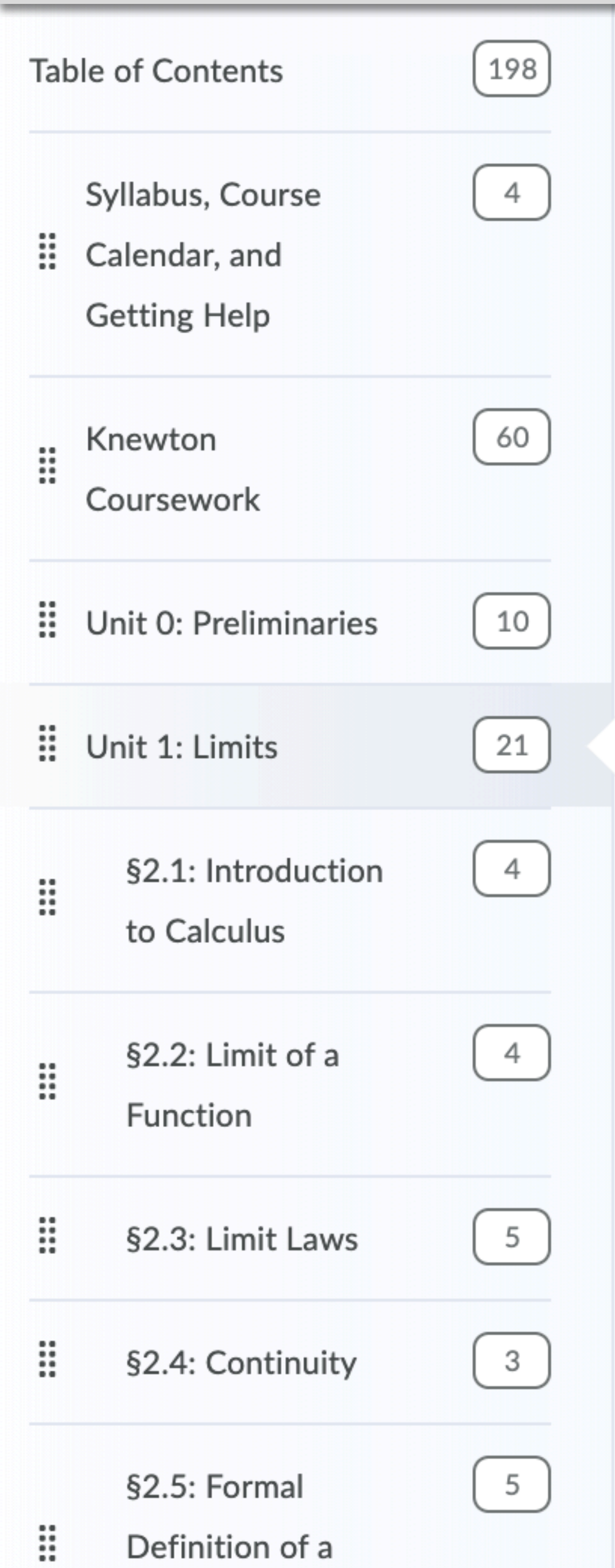
The bulk of the grant work consisted of creating Mathematica applets and compiling the Workbook.

I'll present the **applets** and the **Workbook** in detail after a quick look at the other course materials:

- iCollege **Course Module**
- OpenStax's *Calculus* **textbook**
- Knewton **online assessments**

Course materials

Brightspace/iCollege Module



iCollege is GSU's Brightspace-based online learning platform.

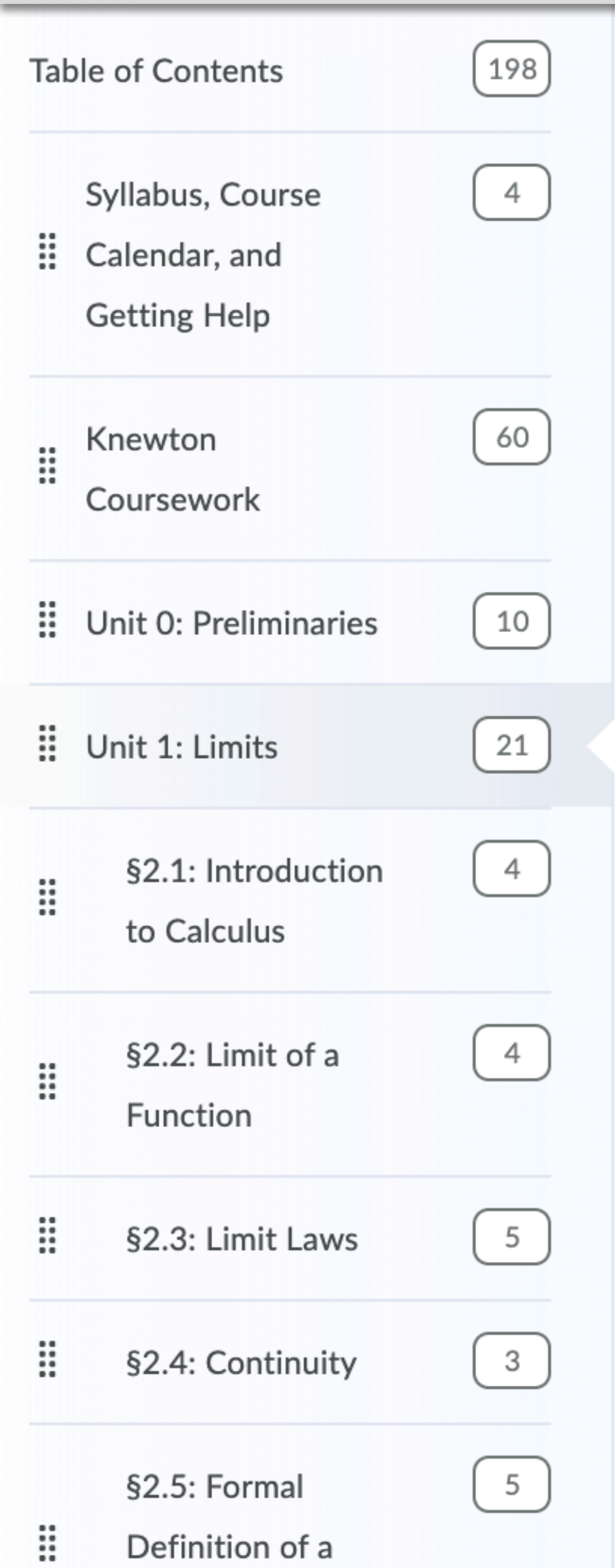
iCollege Course Modules...

- allow all file types
- do not disappear at a corporation's whim
- can be exported (Common Cartridge)

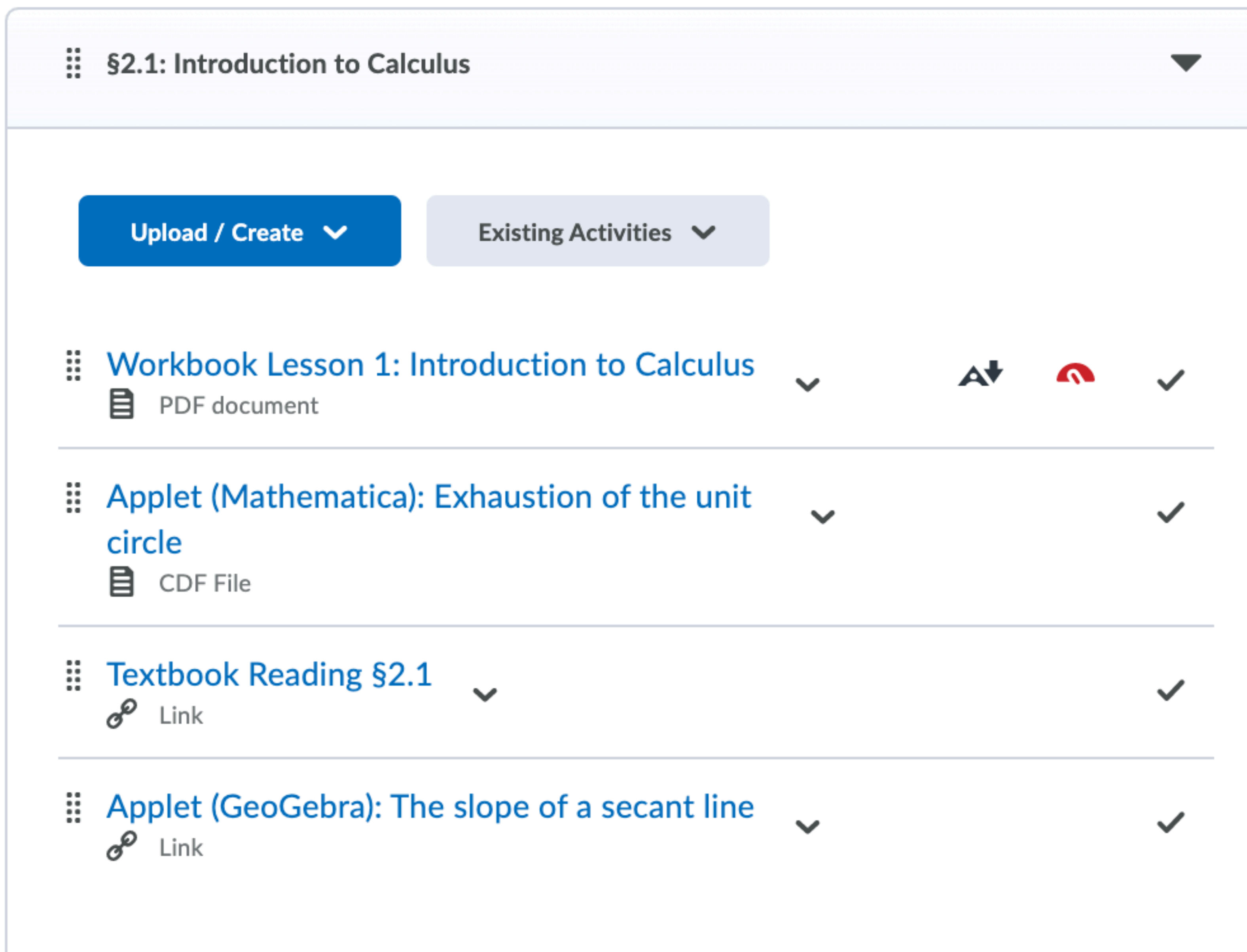
All course materials are accessible to students through the iCollege Course Module.

Course materials

Brightspace/iCollege Module

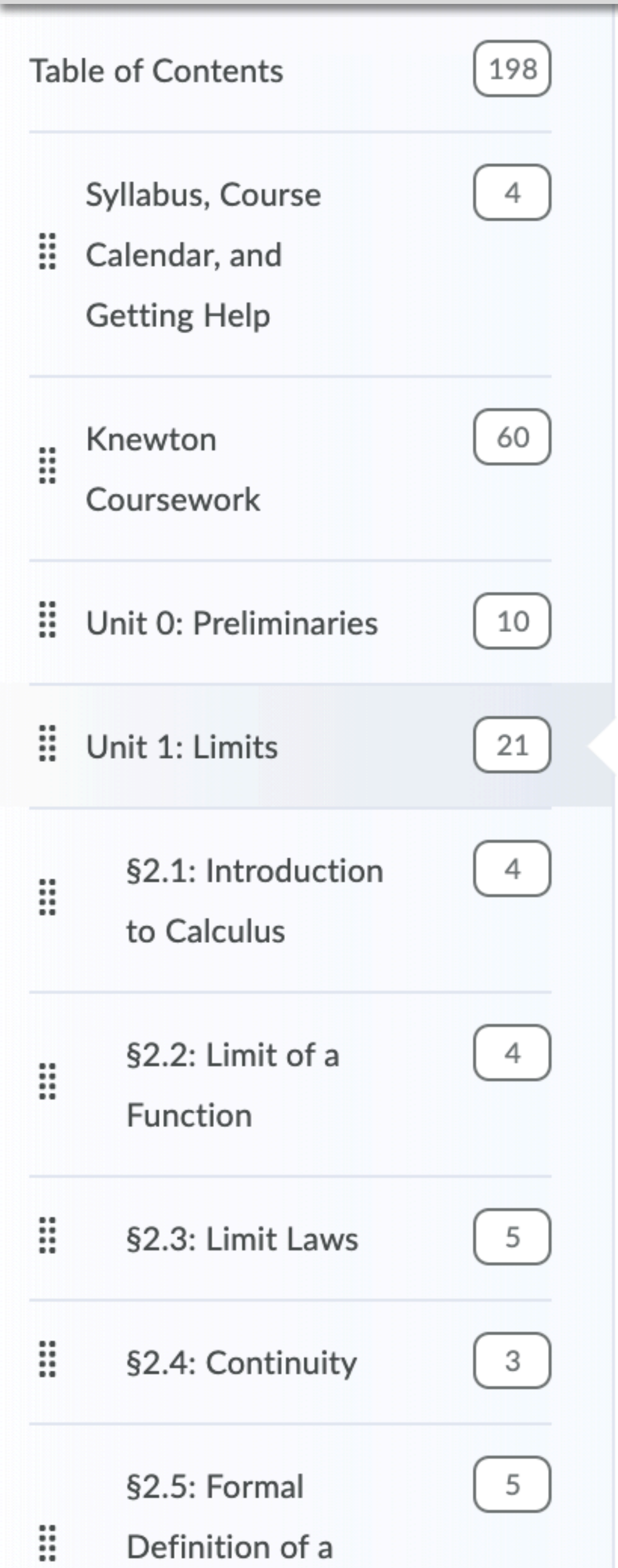


The first "Lesson" in the course is shown below.

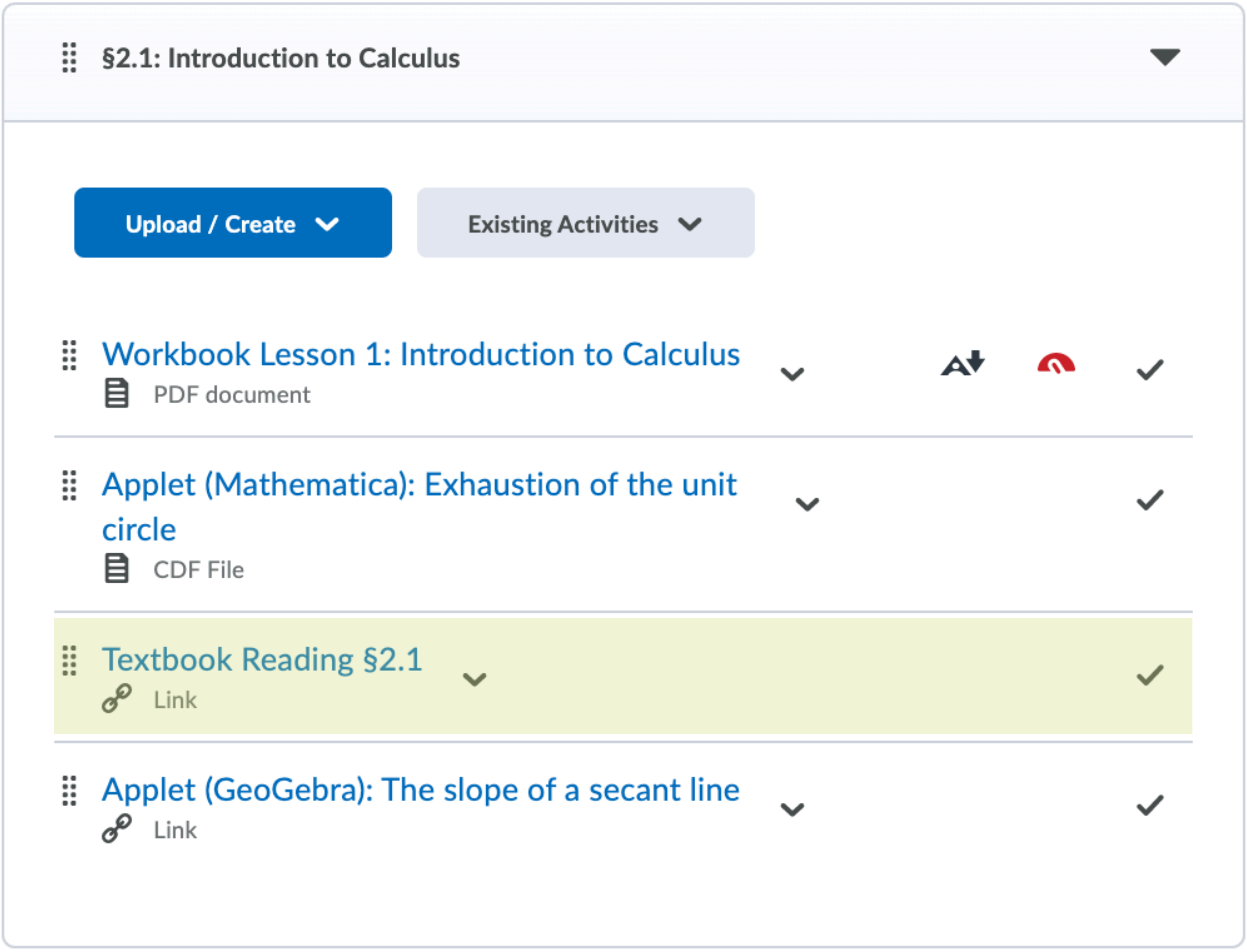


Course materials

Brightspace/iCollege Module

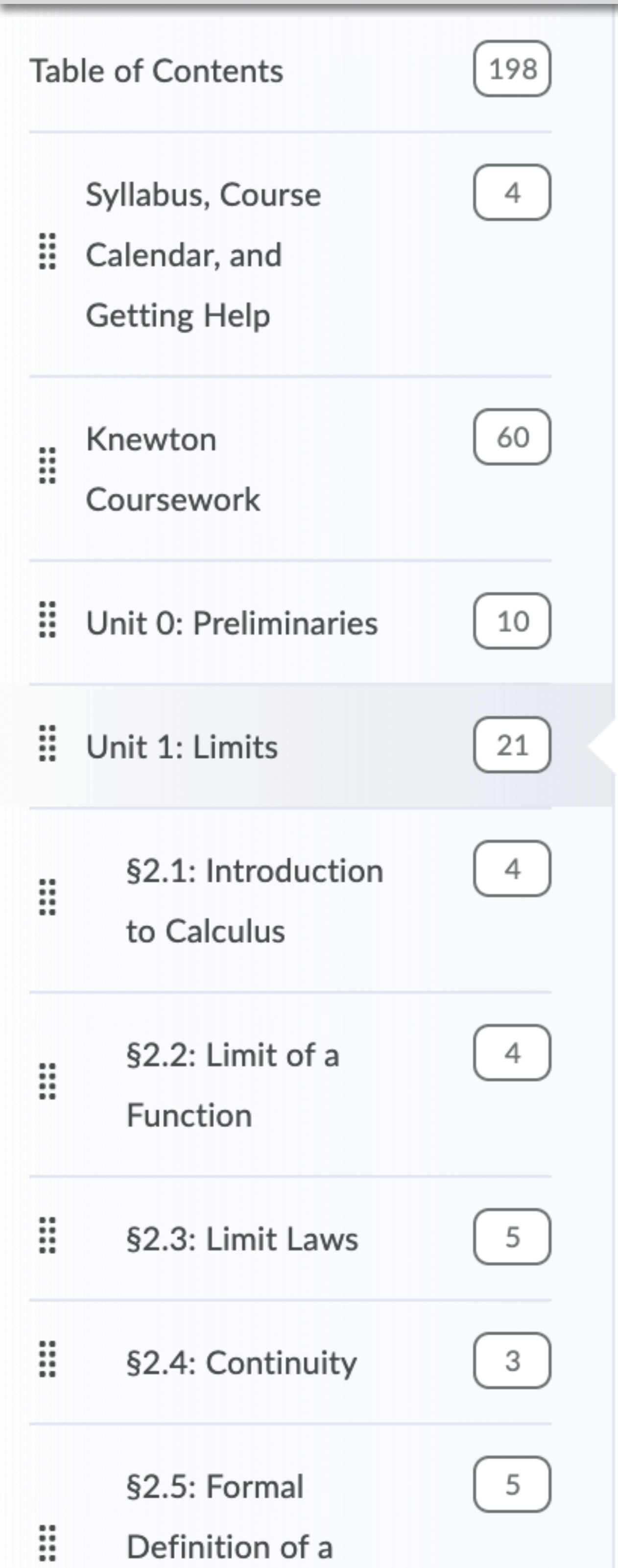


Since the textbook is open source, it's easy to link directly to the reading.

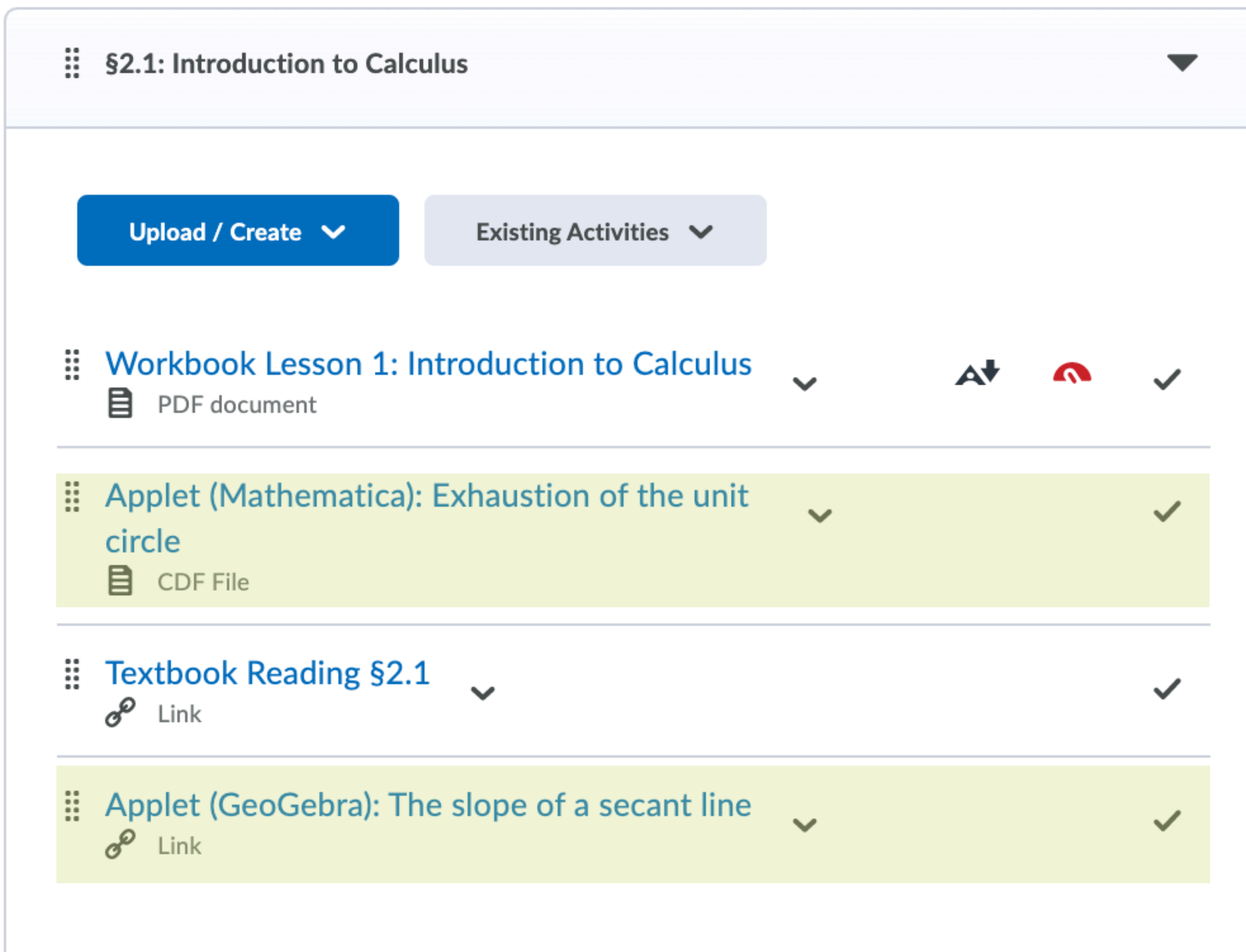


Course materials

Brightspace/iCollege Module



Links to applets I wrote appear in iCollege alongside links to applets hosted outside USG.

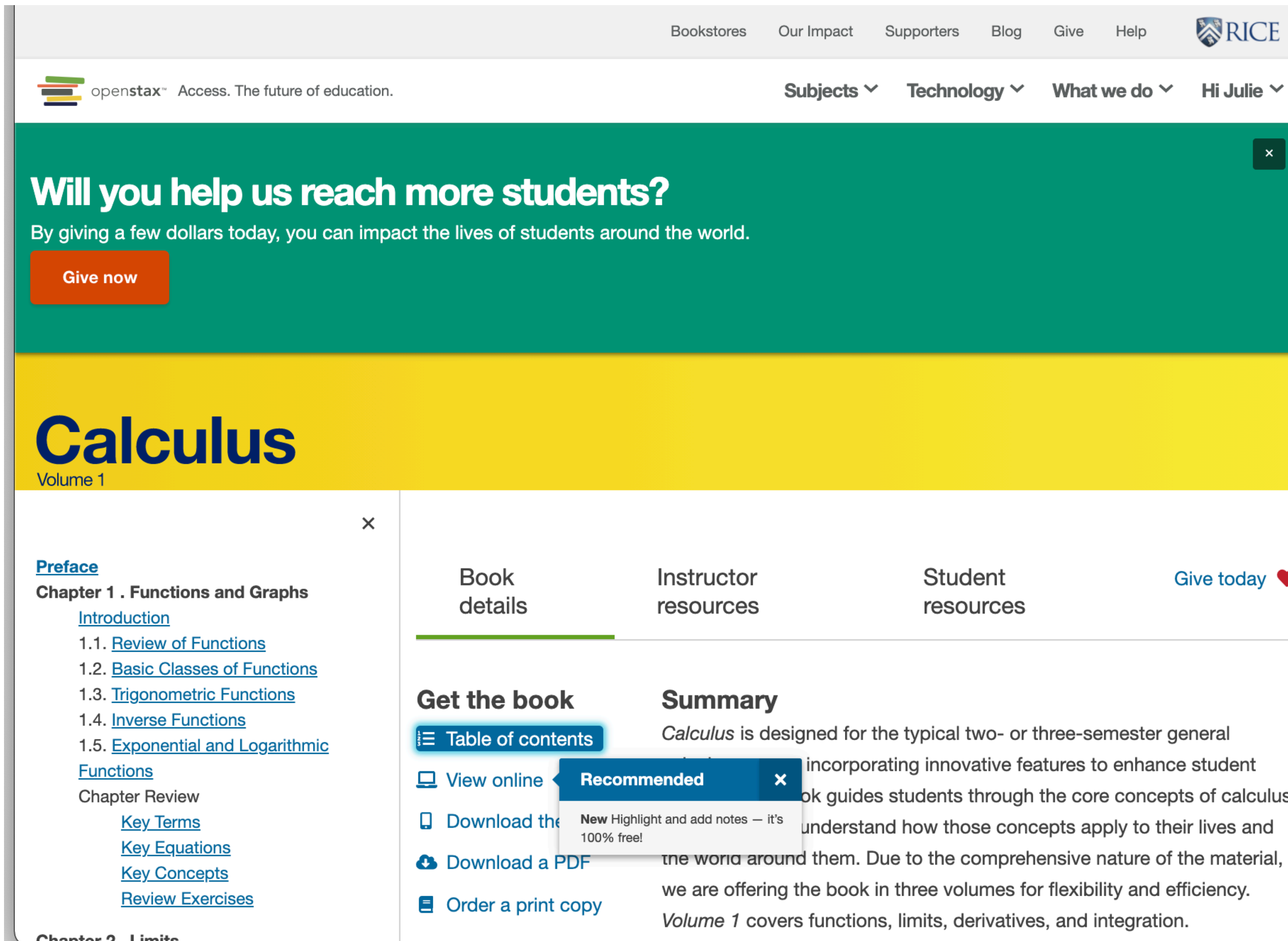


Course materials

OpenStax's *Calculus* textbook

Students can view and add annotations to the textbook for free.

A print copy is also available.



Course materials

OpenStax's *Calculus* textbook

The textbook has adequate exposition and problem sets.

But my students tended to engage at a higher level with the **online homework** than with the textbook.

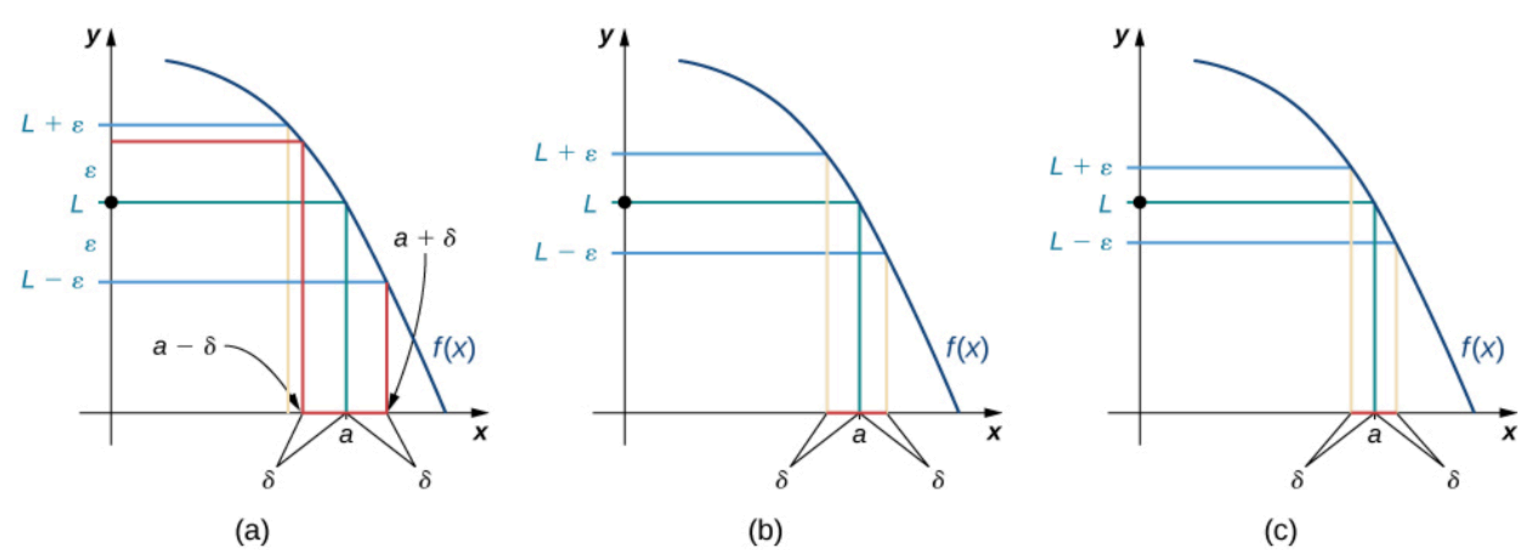
2.5 The Precise Definition of a Limit

$\varepsilon > 0$), an *existential quantifier* (there exists a $\delta > 0$), and, last, a *conditional statement* (if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$). Let's take a look at Table 2.9, which breaks down the definition and translates each part.

Definition	Translation
1. For every $\varepsilon > 0$,	1. For every positive distance ε from L,
2. there exists a $\delta > 0$,	2. There is a positive distance δ from a,
3. such that	3. such that
4. if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$.	4. if x is closer than δ to a and $x \neq a$, then $f(x)$ is closer than ε to L.

Table 2.9 Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. Figure 2.39 shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x)$, a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.



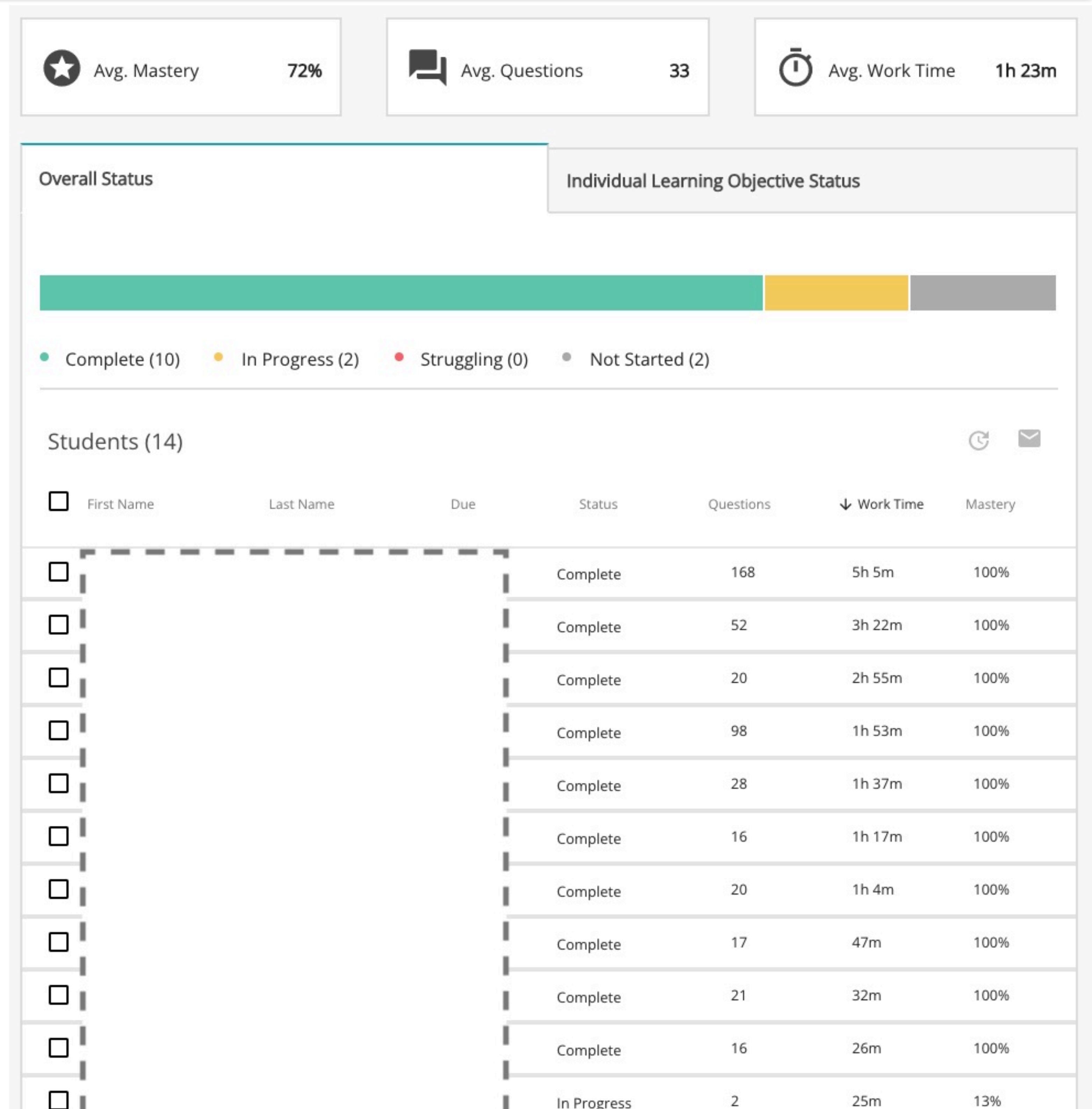
Course materials

Knewton for online assignments

Knewton features **adaptive learning**.

My students love it.

They especially appreciate the **just-in-time review** of topics from earlier math classes.



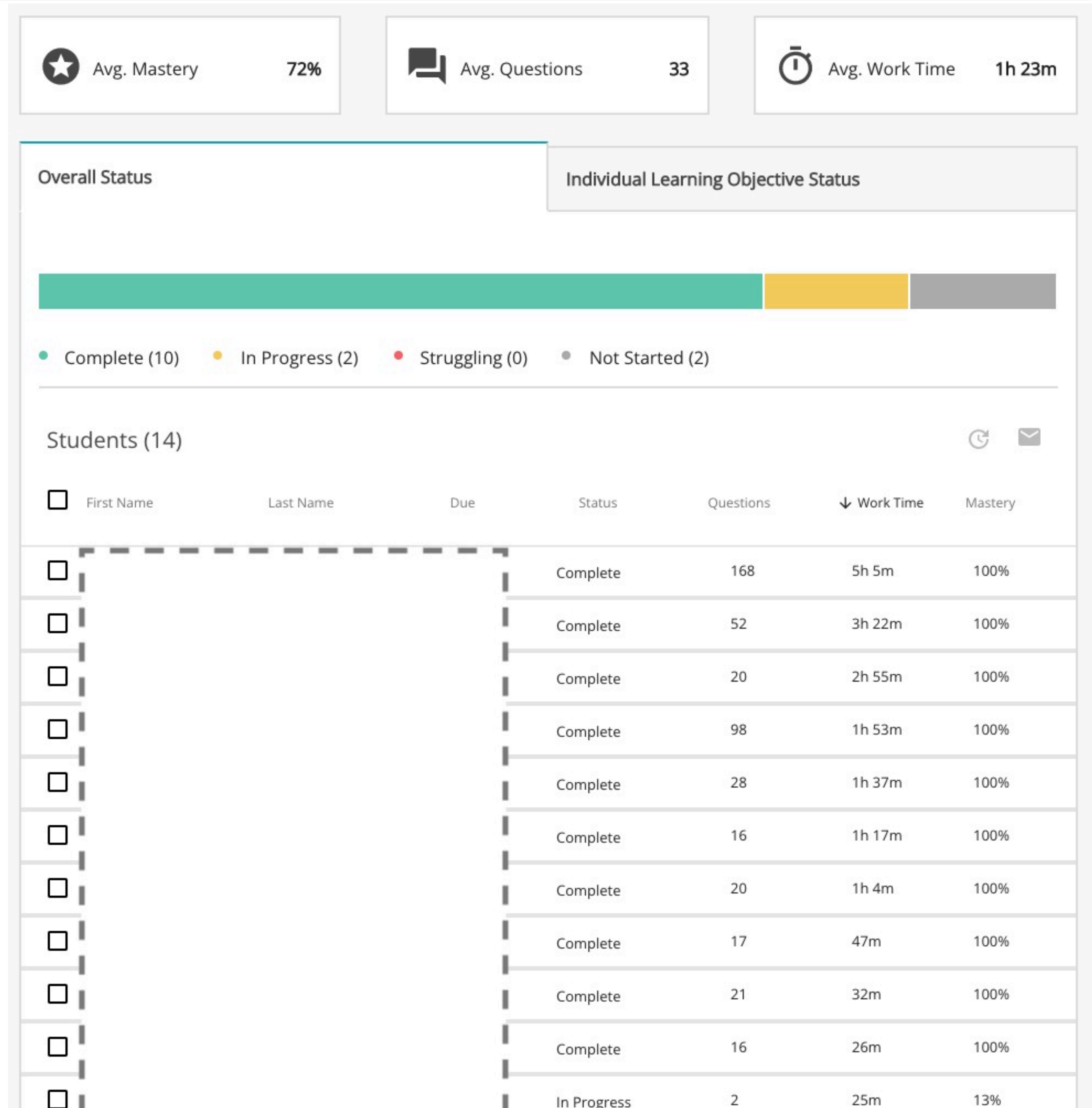
First Name	Last Name	Due	Status	Questions	Work Time	Mastery
			Complete	168	5h 5m	100%
			Complete	52	3h 22m	100%
			Complete	20	2h 55m	100%
			Complete	98	1h 53m	100%
			Complete	28	1h 37m	100%
			Complete	16	1h 17m	100%
			Complete	20	1h 4m	100%
			Complete	17	47m	100%
			Complete	21	32m	100%
			Complete	16	26m	100%
			In Progress	2	25m	13%

Open source Calculus at GSU Dunwoody

Participating Instructors: Kouok Law, Tirtha Timisina, Julie La Corte

A pilot program unrelated to this grant gives free access to Knewton for students in our open source sections of Calculus.

We hope to extend the pilot program to Calculus 2 and 3.



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			Complete	16	1h 17m	100%
			Complete	20	1h 4m	100%
			Complete	17	47m	100%
			Complete	21	32m	100%
			Complete	16	26m	100%
			In Progress	2	25m	13%

Equity means giving people what they need

Like textbooks

Problem:

How many hours will a student need to work in order to pay for their Calculus textbook, assuming they make minimum wage?

Solution:

In this class, the total cost to the student for course materials is **$0**, so the textbook will cost the student **zero hours of labor**, whether they're being paid $7.25 (Federal Fair Labor Standards Act), $5.15 (Georgia minimum wage) or $2.13 (Georgia minimum wage for tipped employees) per hour.

Equity means giving people what they need

Accomodations and administration

Additional equity concerns:

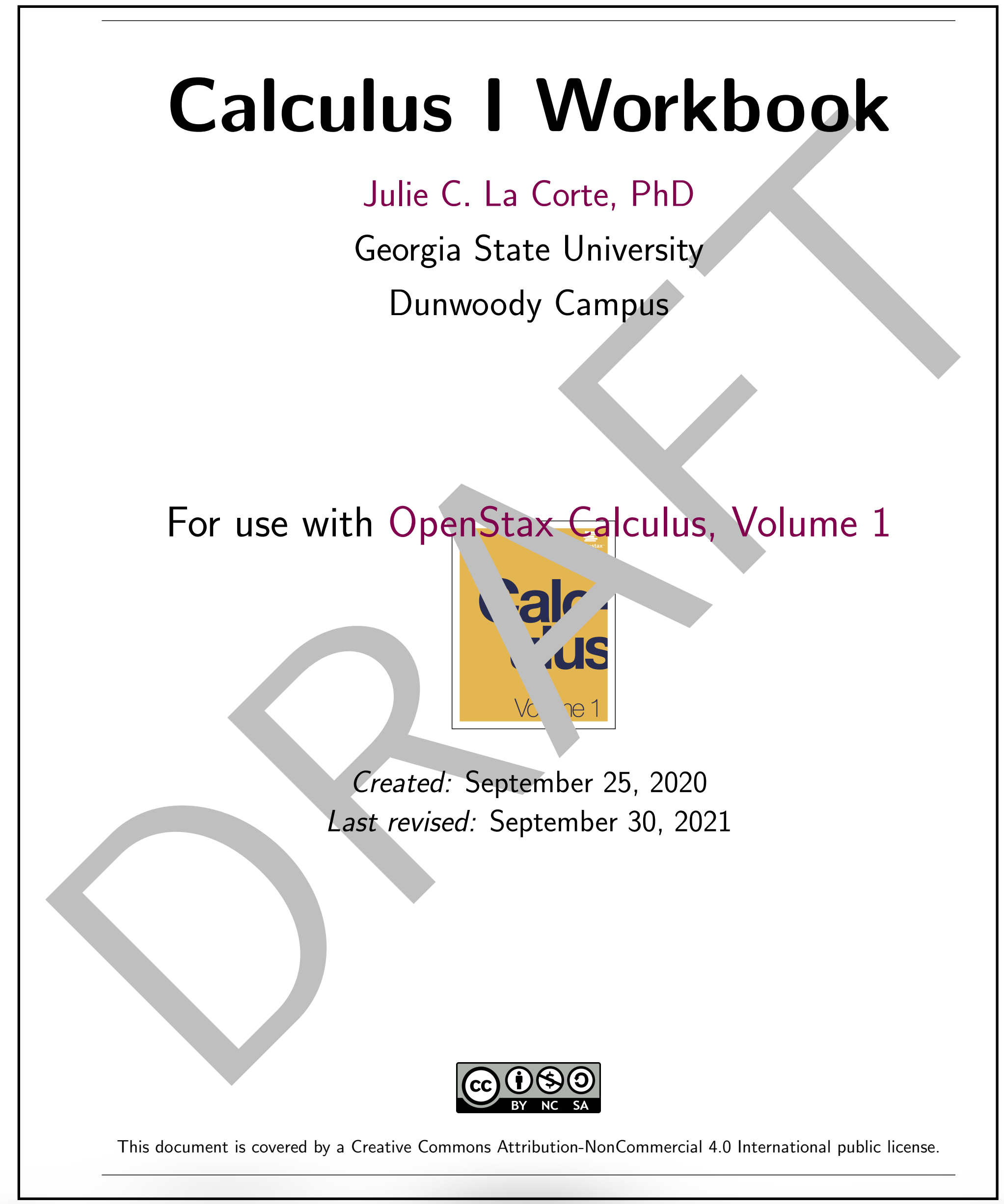
- Access for the visually impaired
- Access to a PC or laptop
- Unplanned-for unknowns

Administrative concerns:

- "Who will I call for technical support?"
- "Can I be compensated for reinventing my course?"

Workbook

- Lessons originally drew on several textbooks
- Future revision will eliminate all material from unknown and/or proprietary sources
- Until all such material is eliminated, the Workbook is still a *draft*


Calculus I Workbook

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Dunwoody Campus

For use with OpenStax Calculus, Volume 1

Created: September 25, 2020
Last revised: September 30, 2021

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DRAFT

Applets

Original applets

Applets written for grant, covering 11 textbook sections:

- Method of Exhaustion
- First Example of a Limit
- Visualizing One-Sided and Two-Sided Limits
- The Limit of $\sin(1/x)$ as $x \to 0$
- Teaching the Definition of the Limit of a Function
- Derivative Sandbox
- Introducing the Chain Rule
- Introducing the Inverse Function Theorem
- Linear Approximation to a Function
- Finding Critical Numbers (*4-in-one*)
- Introducing the Mean Value Theorem
- Finding Intervals of Increase/Decrease (*x4*)
- Applied Optimization Problems (*x3*)

Applets

Original applets, categorized by intended use

Guided discussion:

- Method of Exhaustion
- First Example of a Limit
- Visualizing One-Sided and Two-Sided Limits
- Introducing the Chain Rule
- Introducing the Inverse Function Theorem
- Linear Approximation to a Function
- Introducing the Mean Value Theorem

Several of the applets were intended for in-class use.

The Workbook typically includes still screenshots of each applet.

Interested students can animate and manipulate the pictures in the Workbook by exploring the applets.

Applets

Original applets, categorized by intended use

Open exploration:

- Derivative Sandbox

The "Derivative Sandbox" encourages free exploration.

Using this applet, students tended to discover the relation between turning points and zeros of f' on their own.

Self-guided activities:

- Teaching the Definition of the Limit of a Function
- The Limit of $\sin(1/x)$ as $x \to 0$

Applets

Original applets, categorized by intended use

Open exploration:

- Derivative Sandbox

Self-guided activities:

- Teaching the Definition of the Limit of a Function
- The Limit of $\sin(1/x)$ as $x \to 0$

Other applets display explicit questions.

For example, one applet asks the student to find a $\delta > 0$ satisfying the "ε challenge" in the definition of the limit of $f(x)$ as $x \to a$.

The values of a, δ, and ε can all be manipulated by the student using sliders.

Applets

Original applets, categorized by intended use

Strategy guides:

- Finding Critical Numbers (*4-in-one*)
- Finding Intervals of Increase/Decrease (*x4*)
- Applied Optimization Problems (*x3*)

A final set of applets guides the student through multipart problems, emphasizing the "Big Picture" strategy.

For instance, after students have found the critical numbers of several functions by hand, they are then directed to the applets, where the steps are presented visually without calculations.

Applets

Original applets, categorized by intended use

Strategy guides:

- Finding Critical Numbers (*4-in-one*)
- Finding Intervals of Increase/Decrease (*x4*)
- Applied Optimization Problems (*x3*)

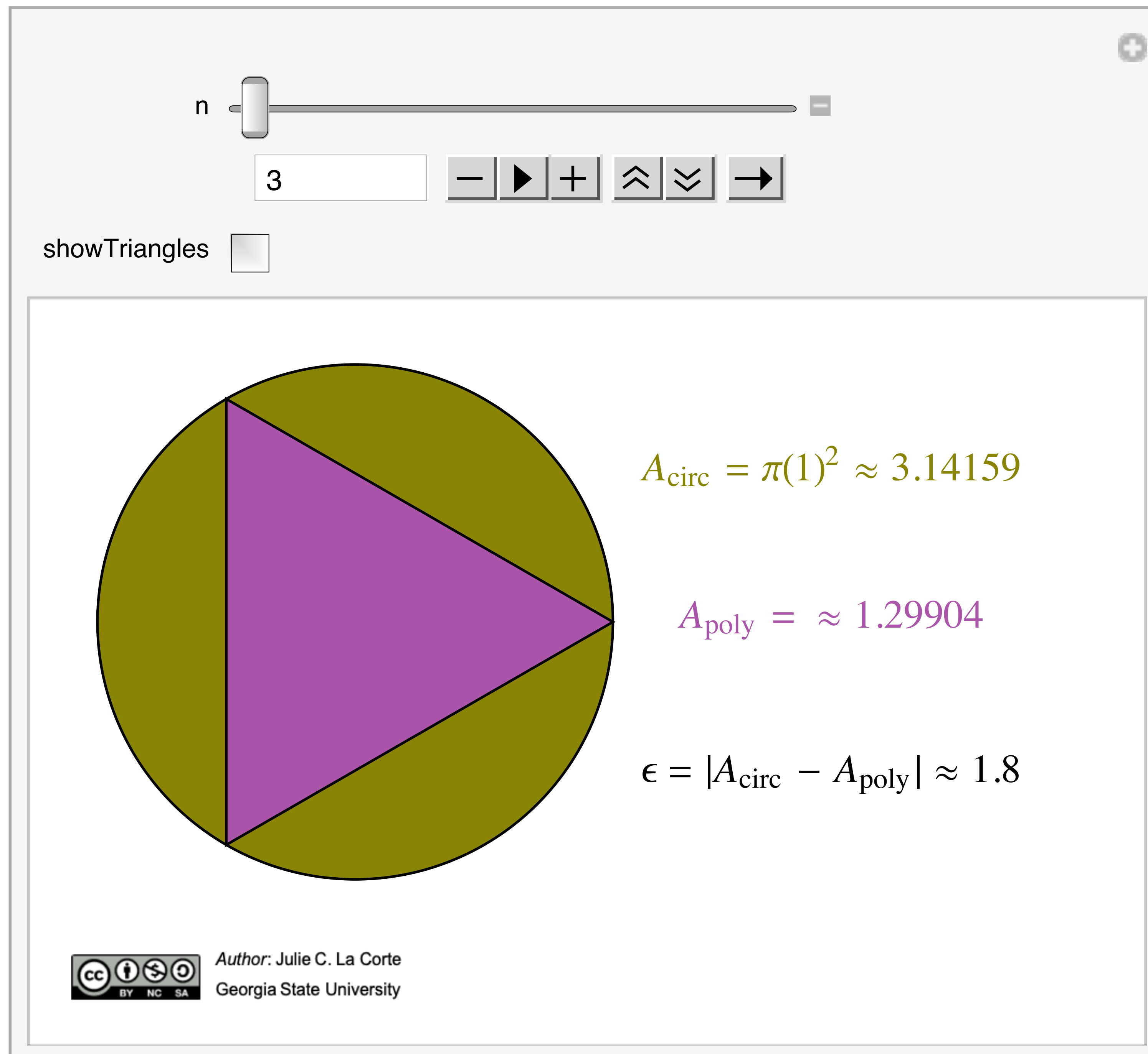
Students report that...

- these **"strategy guide"** applets help them practice the specific exercises whose solutions appear in the Workbook, and
- these applets remind them how to organize their work when working different exercises of the same type.

Applets

1. Method of Exhaustion

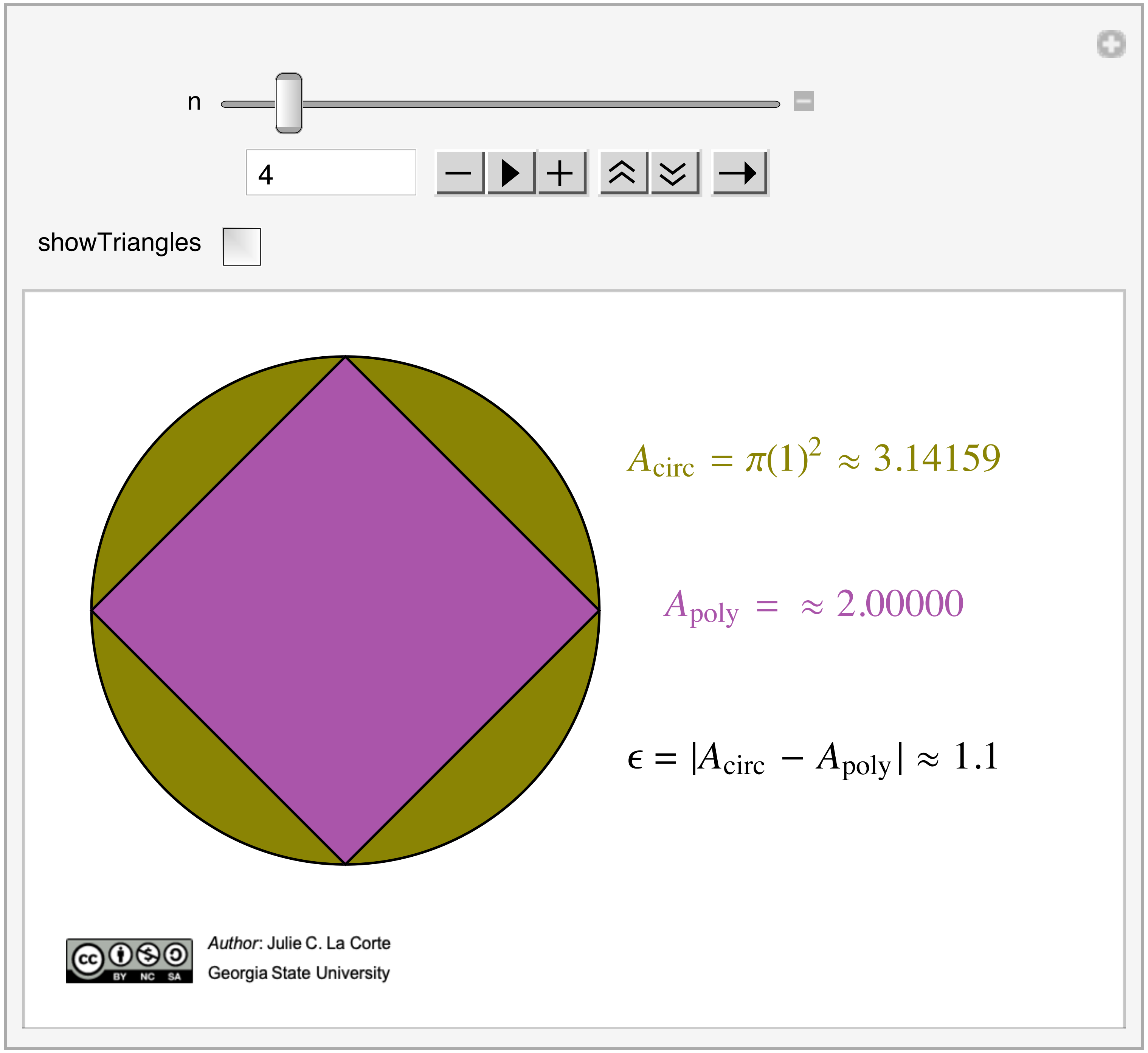
The first section in the OpenStax textbook (and the Workbook) is a breezy tour of the limiting processes encountered in Calculus 1.



Applets

1. Method of Exhaustion

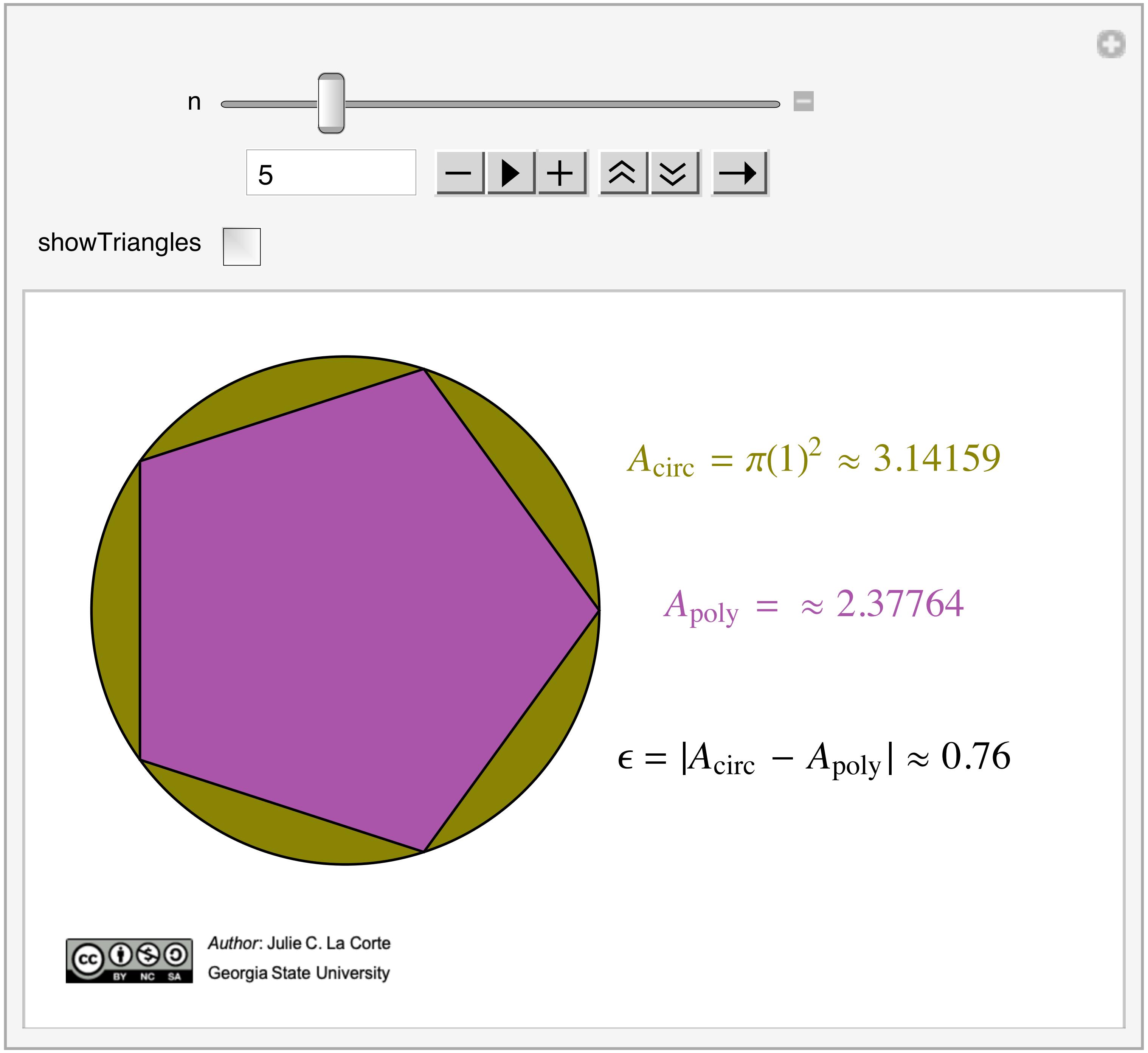
The first example of a limiting process my students see is the classical problem of exhausting the area of a circle by an inscribed regular *n*-gon.



Applets

1. Method of Exhaustion

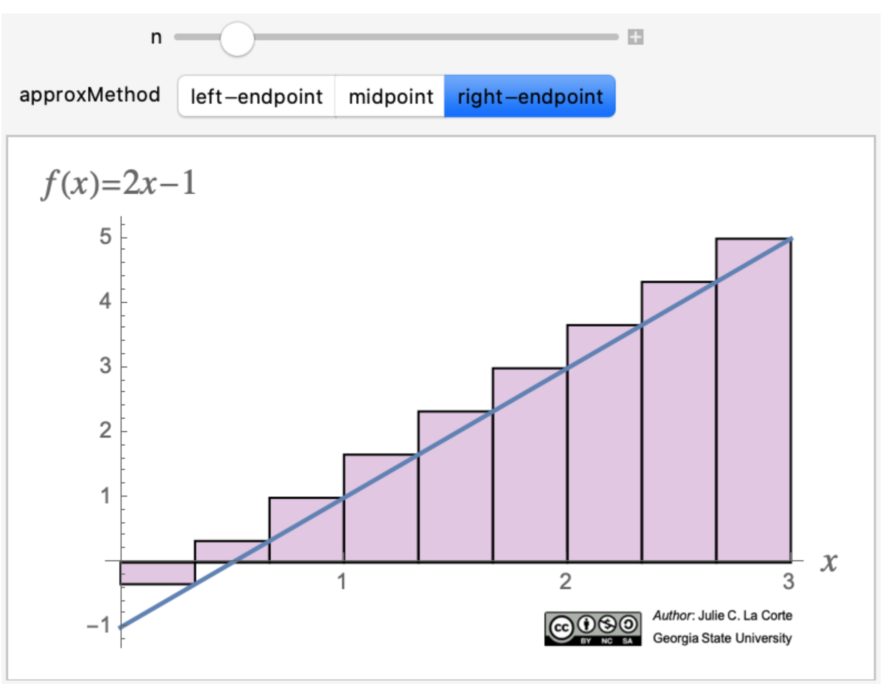
Within the first few minutes of class, students are thus exposed to the idea of *a process that can be extended indefinitely, with an associated error term that approaches 0.*

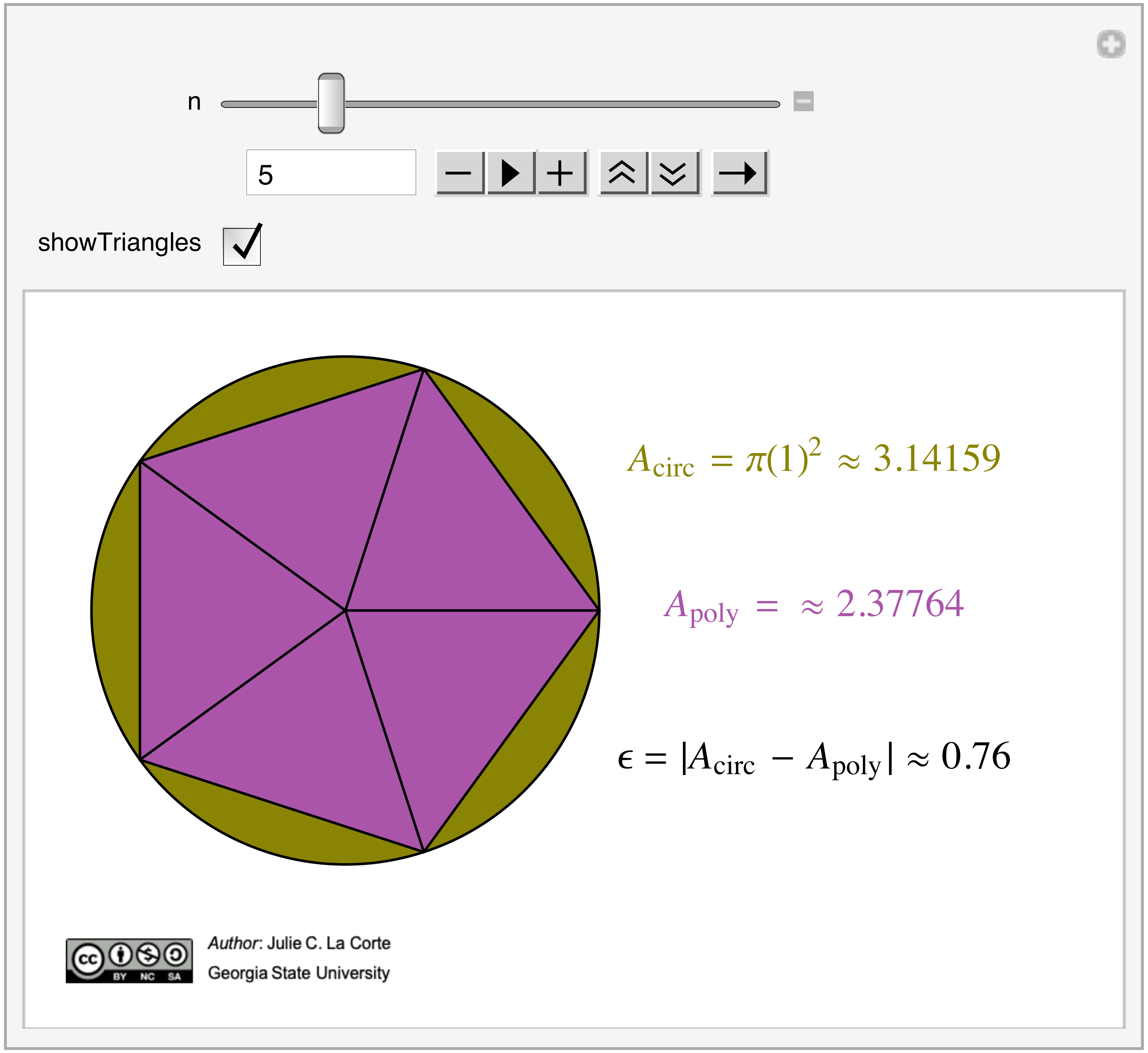


Applets

1. Method of Exhaustion

The decomposition of the *n*-gon into triangles obviously foreshadows Riemann sums.

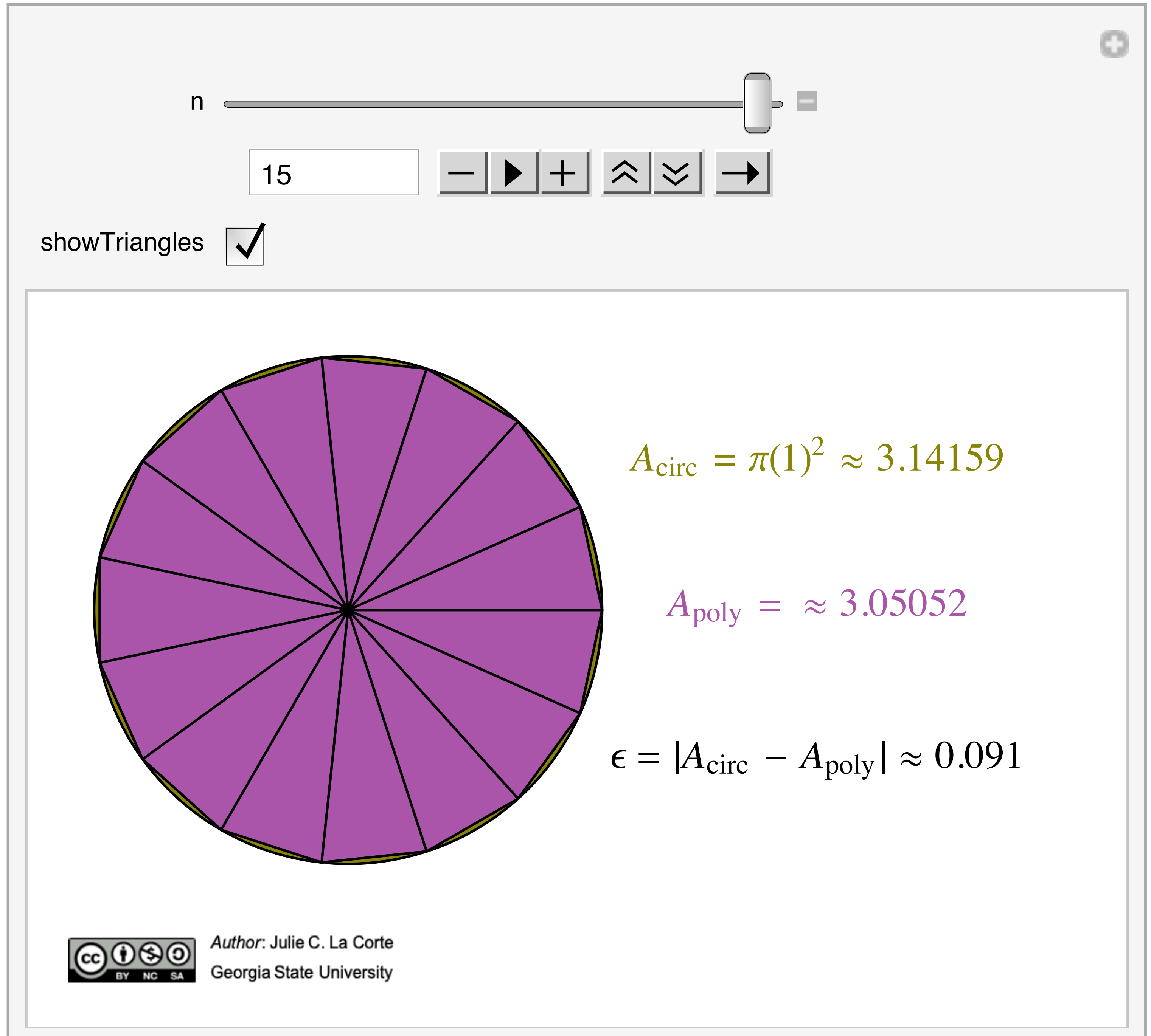




1. Method of Exhaustion

The decomposition of the *n*-gon into triangles obviously foreshadows Riemann sums.

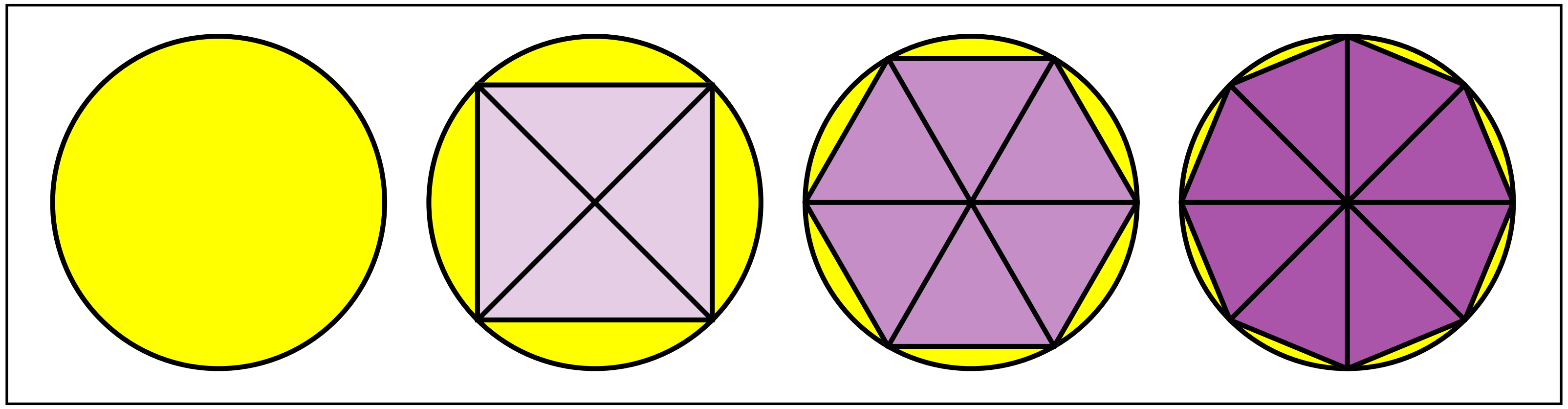
The first section in the OpenStax textbook makes this foreshadowing explicit.

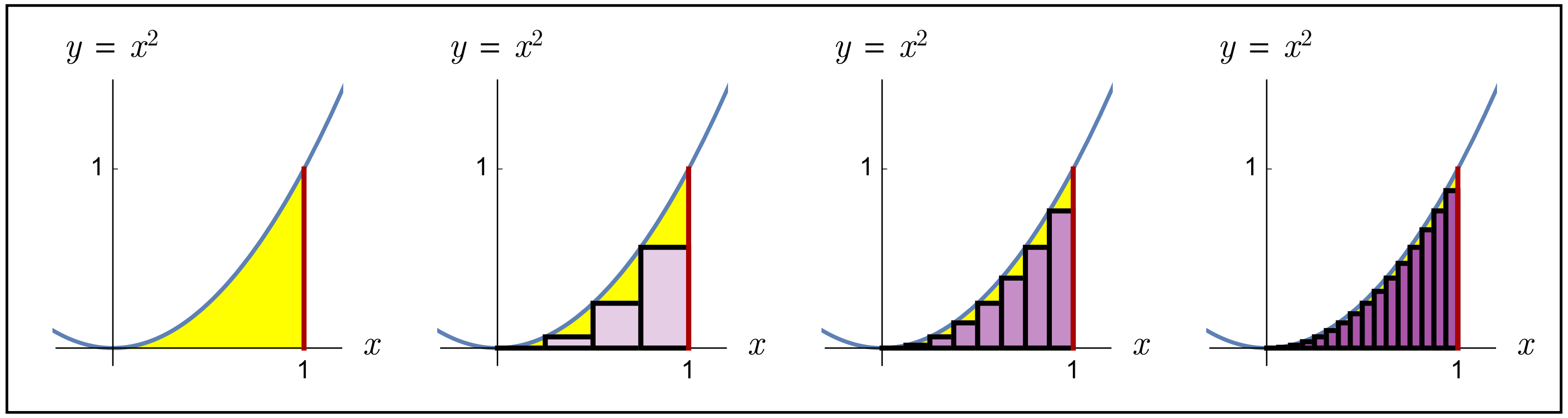


Applets

1. Method of Exhaustion

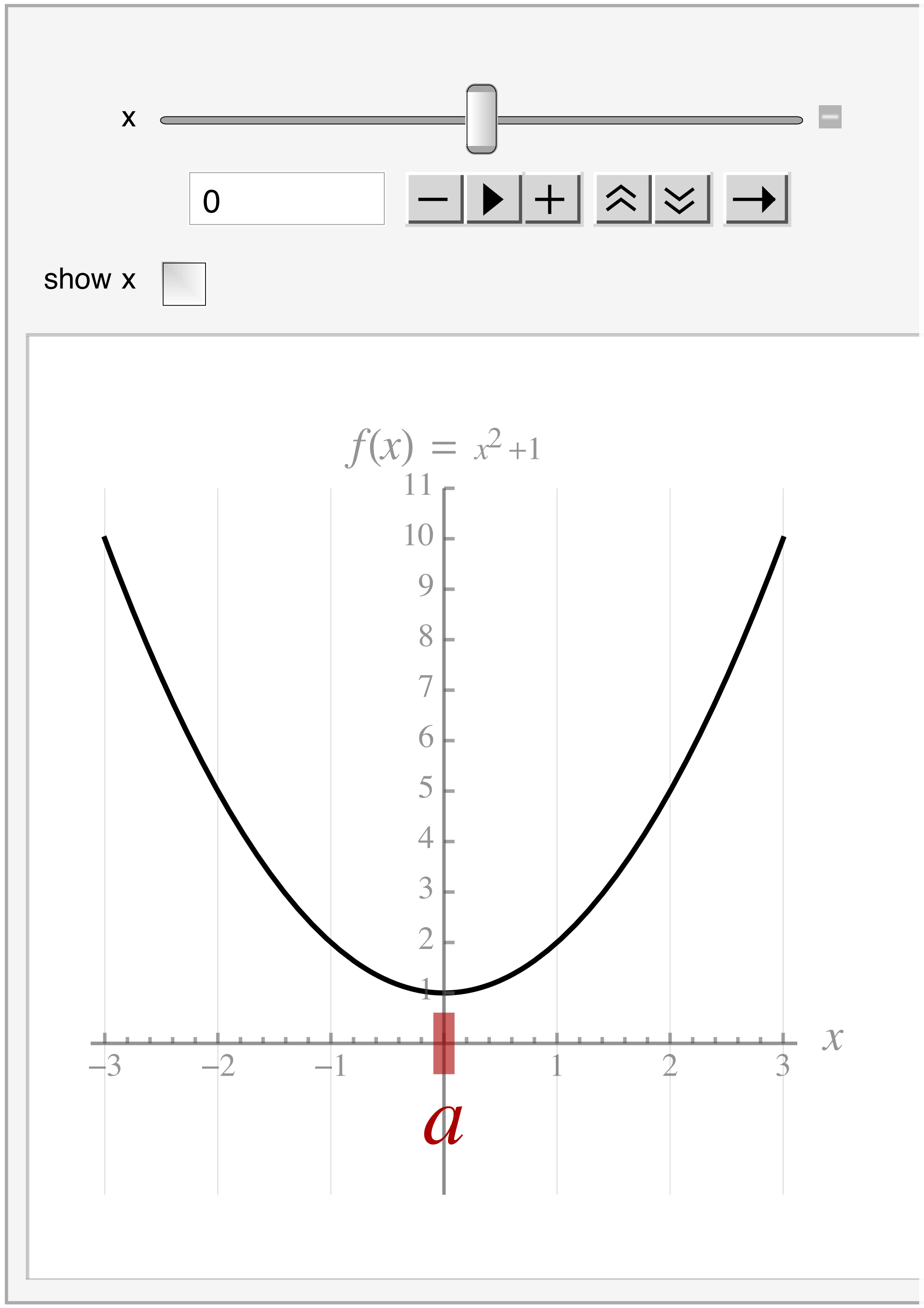
The Workbook contains screenshots of the animations which students see during lecture.





Applets

2. Determining the Limit of a Function from its Graph

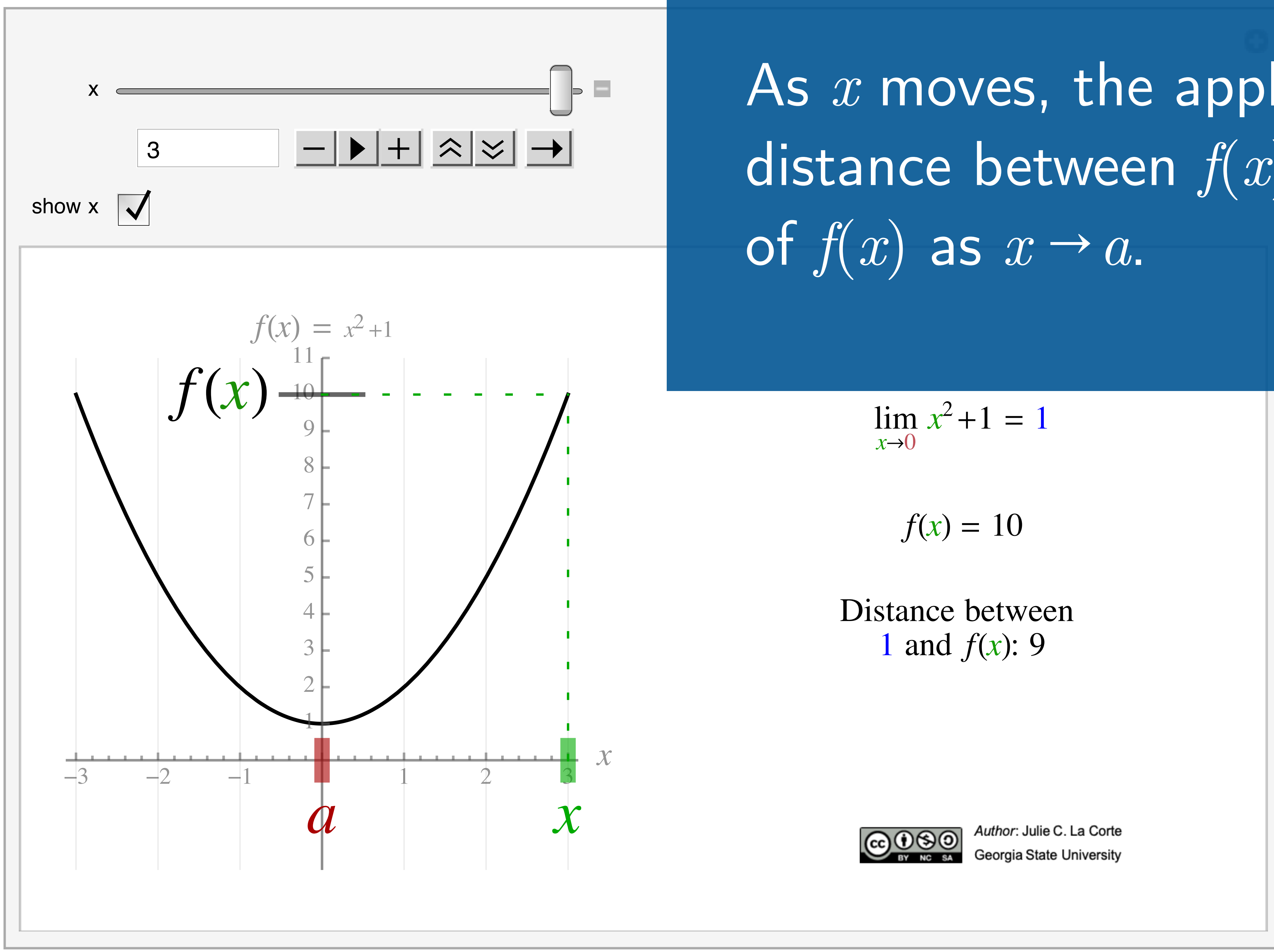


Finding a limit graphically was a known trouble spot for our students.

This applet allows the student and/or instructor to move x using the slider.

Applets

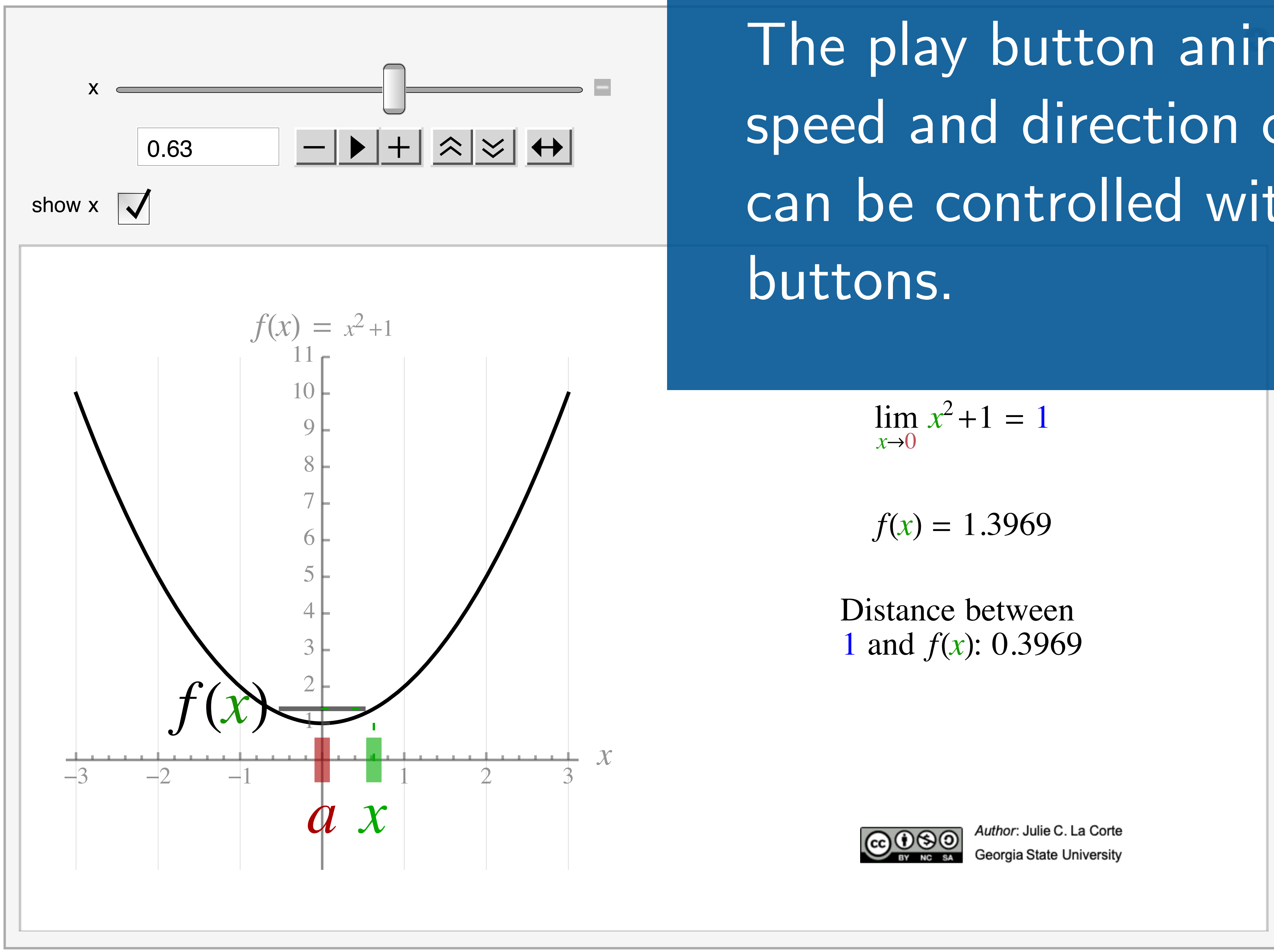
2. Determining the Limit of a Function from its Graph



As x moves, the applet updates the distance between $f(x)$ and the limit of $f(x)$ as $x \to a$.

Applets

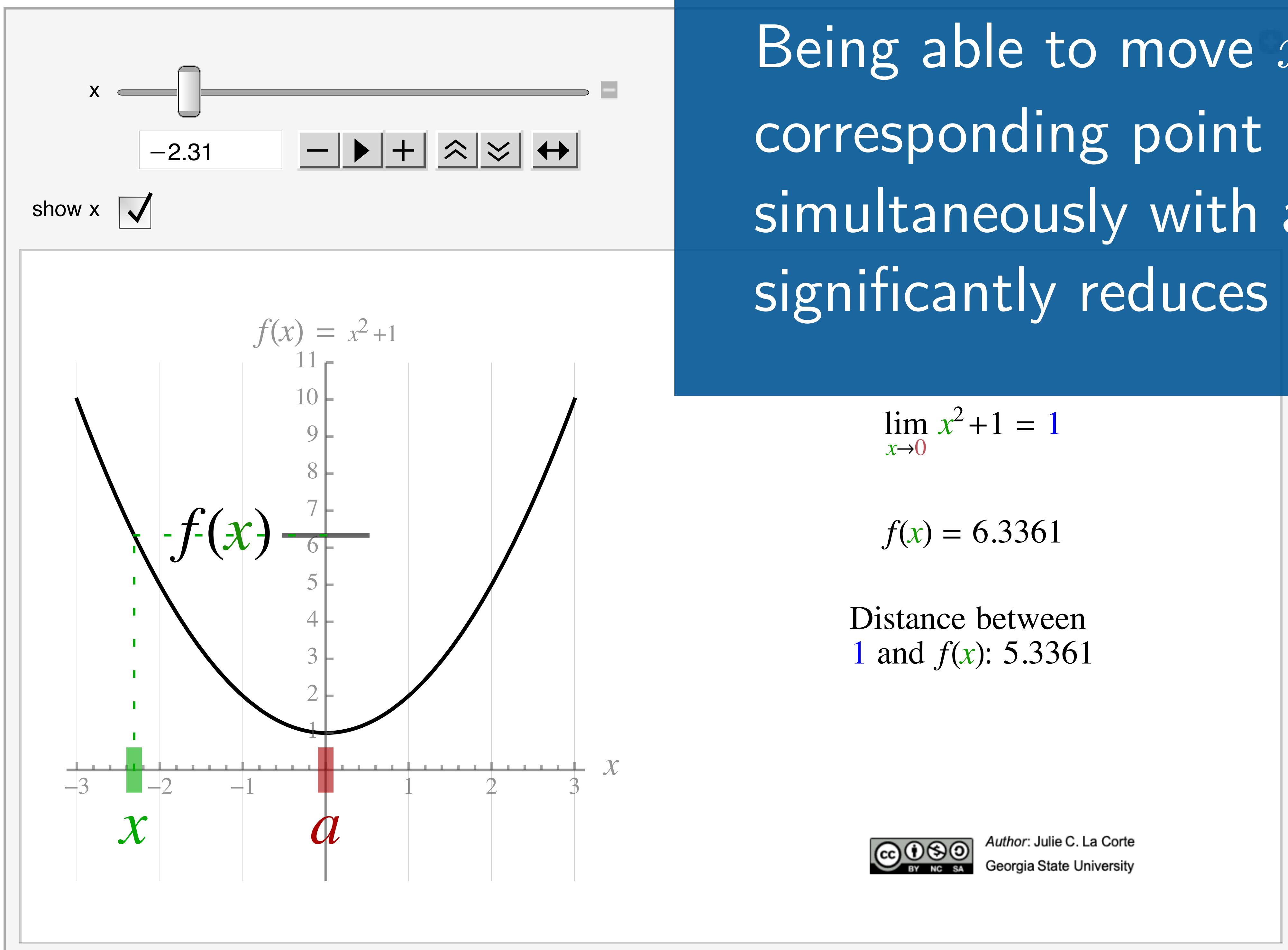
2. Determining the Limit of a Function from its Graph



The play button animates x. The speed and direction of animation can be controlled with other buttons.

Applets

2. Determining the Limit of a Function from its Graph



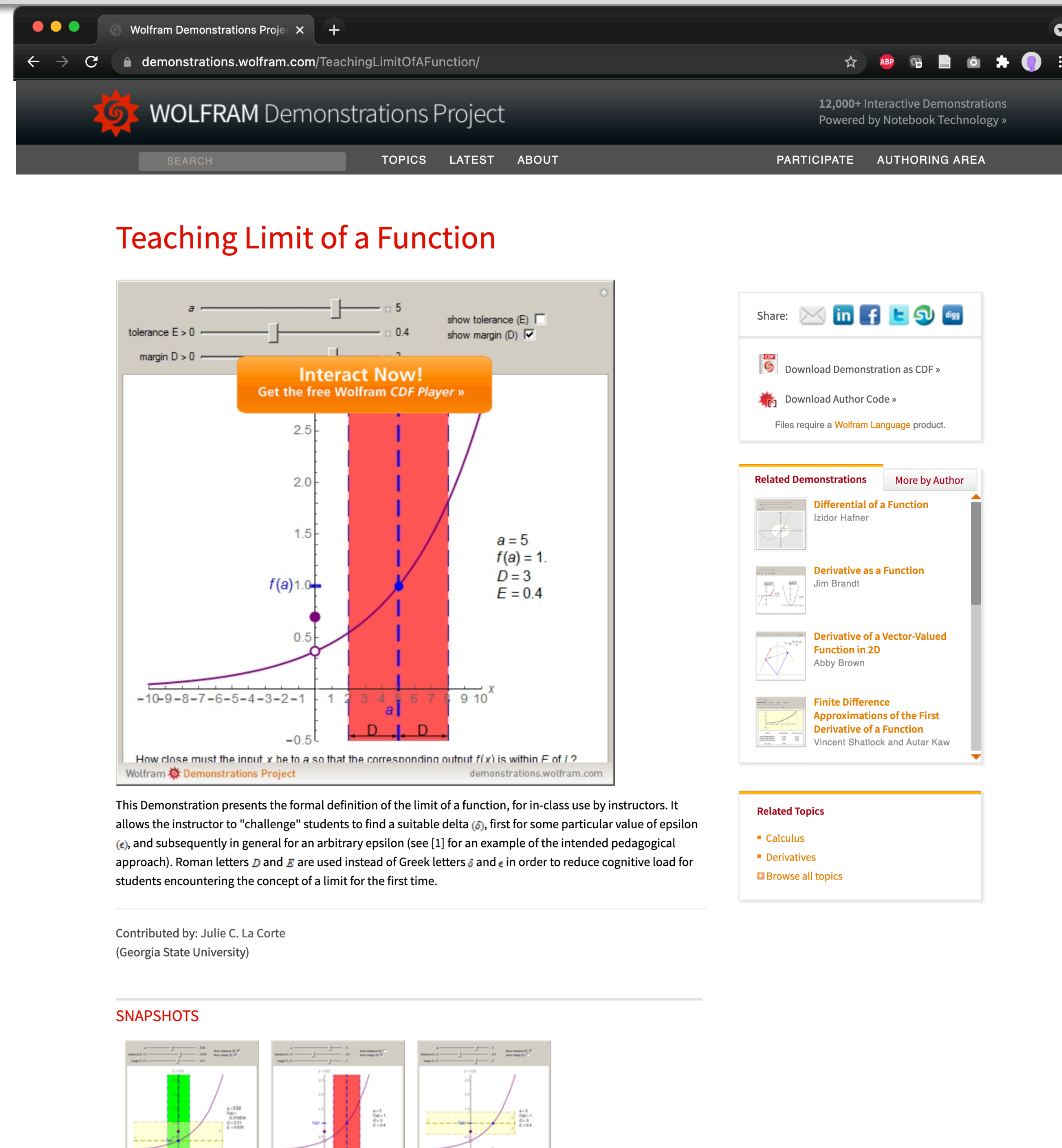
Being able to move x and the corresponding point $(x, f(x))$ simultaneously with a slider significantly reduces handwaving.

Applets

Downloading from Wolfram Demonstrations Project

The first applet students were intended to download and experiment with on their own is published on Wolfram's website.

2–5 Teaching the Definition of the Limit of a Function.nb



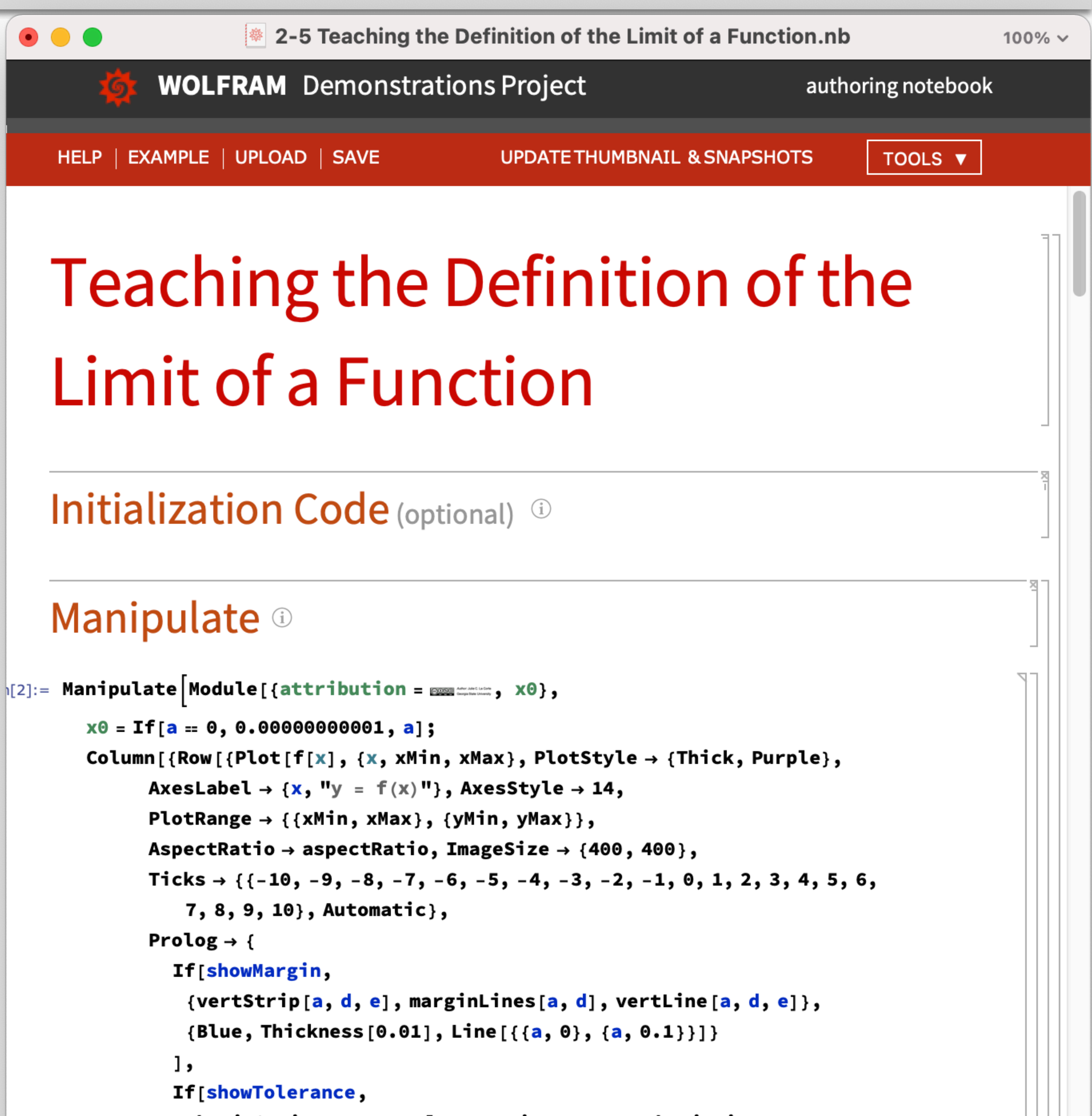
Applets

Downloading from Wolfram Demonstrations Project

The ".NB" file is the source code.

Mathematica is required to edit it.

2-5 Teaching the Definition of the Limit of a Function.nb

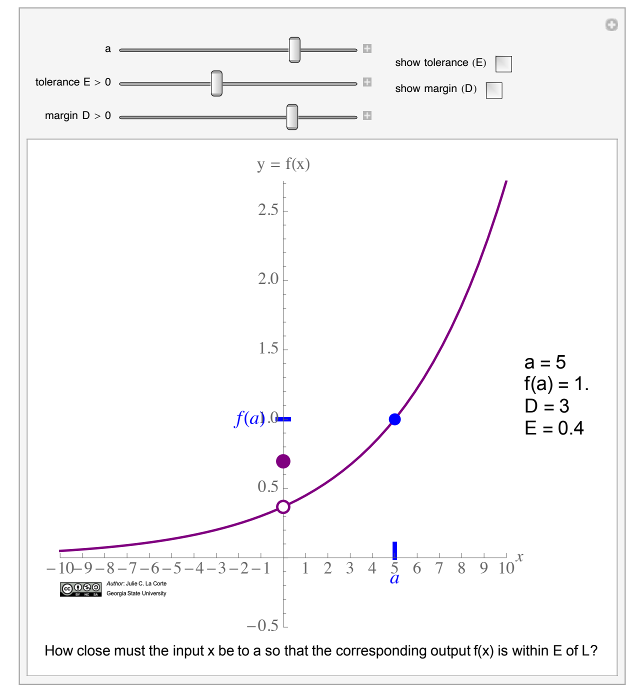


Applets

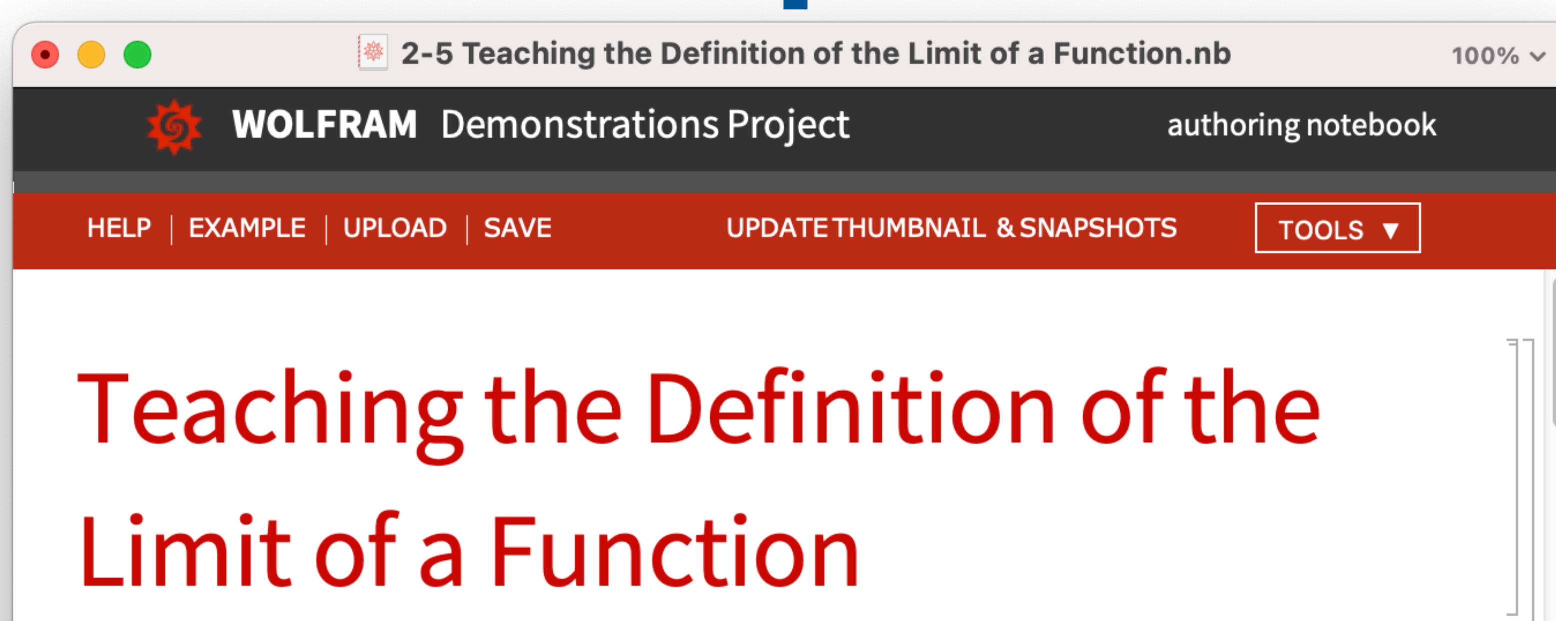
Downloading from Wolfram Demonstrations Project

Students can download a ".CDF" version (which requires only a free player) through iCollege.

2–5 Teaching the Definition of the Limit of a Function.cdf



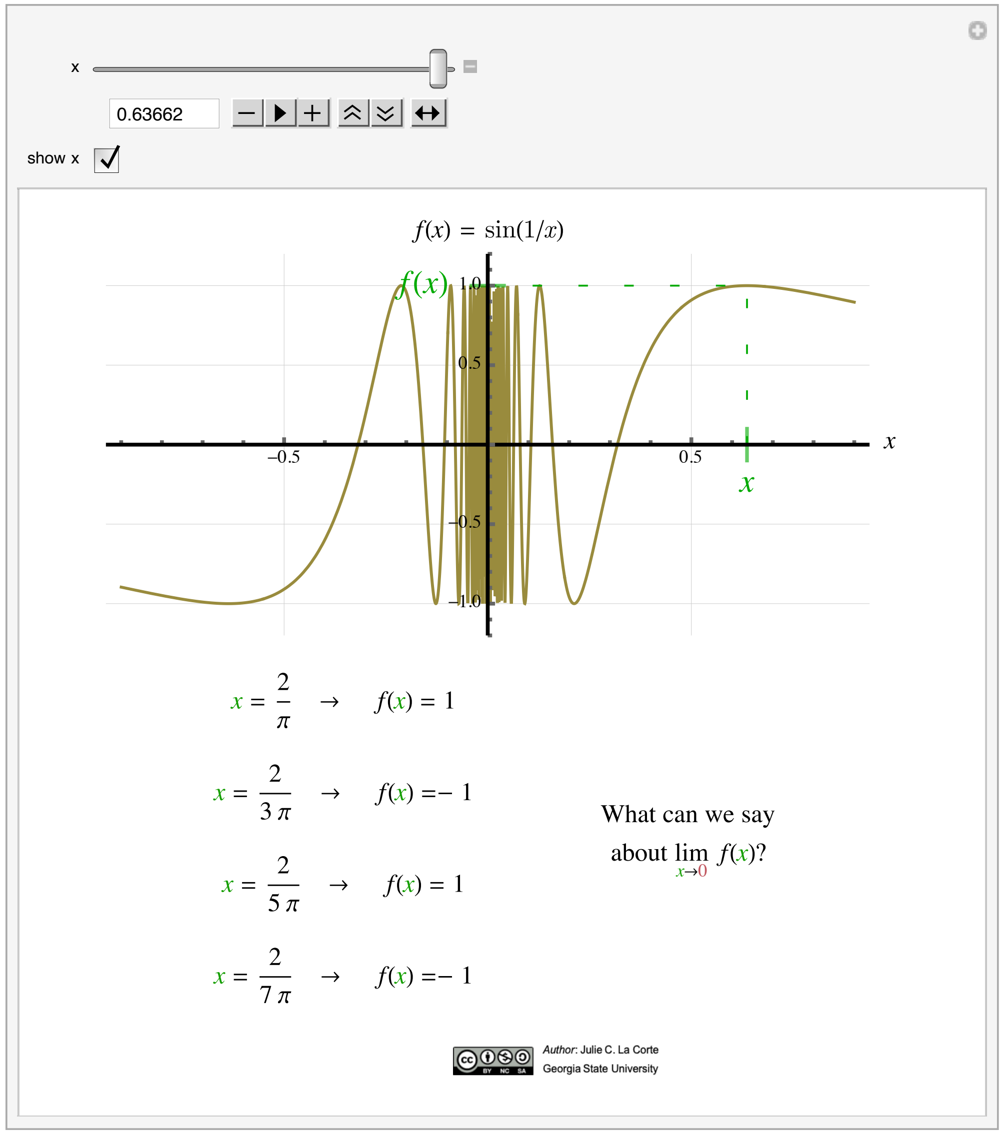
2–5 Teaching the Definition of the Limit of a Function.nb

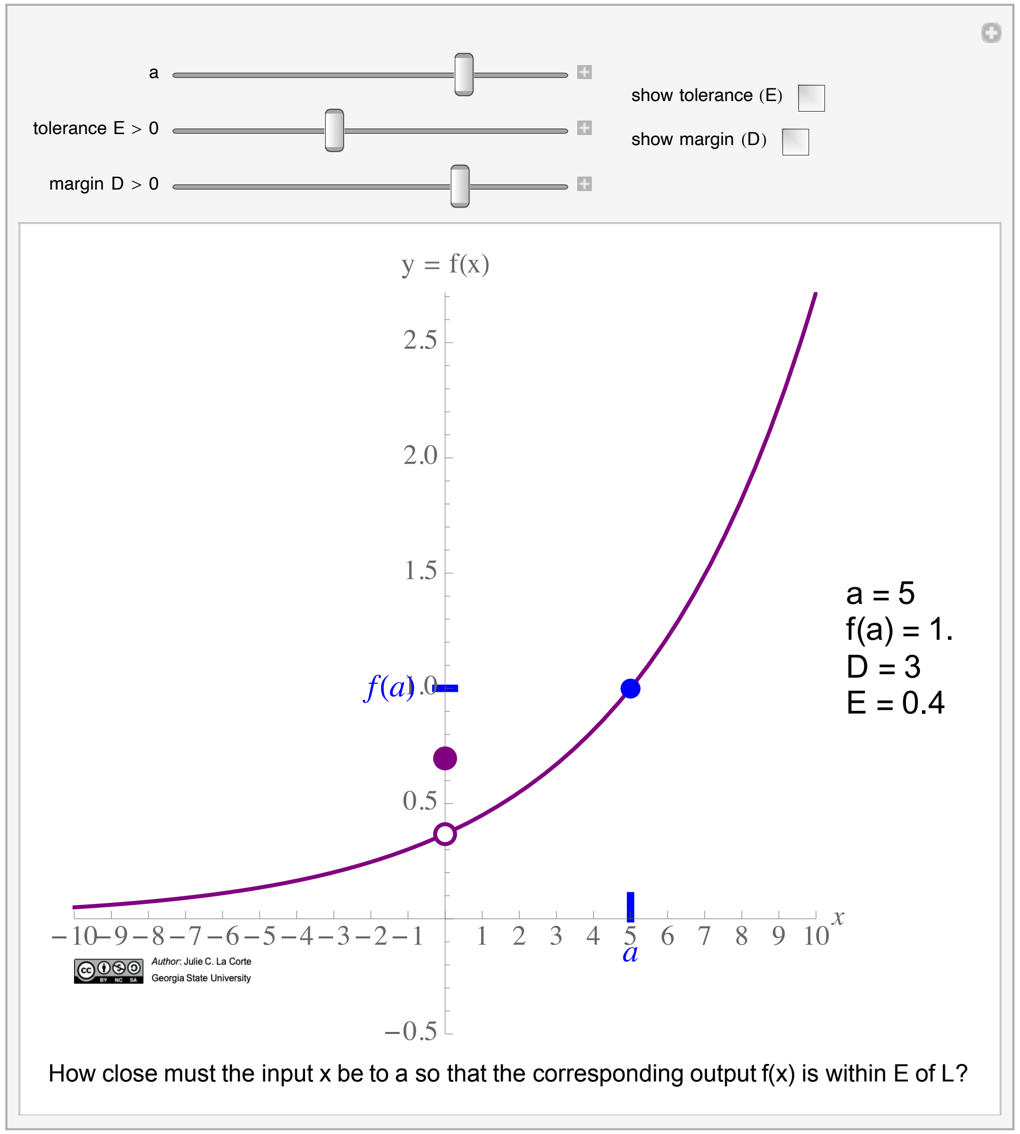


Applets

Formal Definition of the Limit of a Function

The Workbook lesson on the formal (ε-δ) definition of the limit relies heavily on applets.





Applets

Formal Definition of the Limit of a Function

In a face-to-face class, I'd begin the lesson by illustrating how to interpret an inequality of the form

$$|w - a| < c$$

using a number line and a piece of string.

Workbook Lesson 5

§2.5, The Epsilon-Delta Definition of a Limit

Last revised: 2021-06-15 07:33

Objectives

- Interpret an inequality of the form $0 < |x - a| < c$ as a statement about the distance between x and a.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist. *(Moved from Lesson 2, §2.2)*
- Describe the idea behind the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

Inequalities representing distance

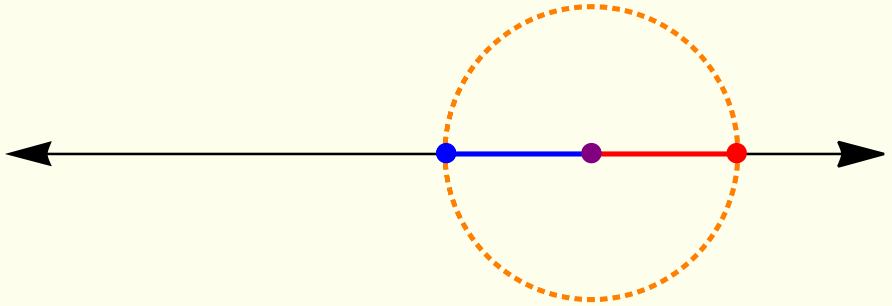
The **distance** between two numbers w and a is $|w - a| \geq 0$.

Let $c > 0$. The inequality

$$|w - a| < c$$

means that the distance $|w - a|$ between w and a is less than c. (We use the absolute value bars because the **difference** $w - a$ might be negative, while distance is by definition never negative.)

Anchoring one end of a piece of string at the purple point on the number line below, pinch off a length of string—call the length c—and swing it around the purple point like a compass to see why c is sometimes called the "radius" of the inequality.



Applets

Formal Definition of the Limit of a Function

"Hybridizing" the Lesson meant asking the student to do for themselves what I might not be physically present to do for them.

The text (in red) prompts the student to build their inuition using string.

Workbook Lesson 5

§2.5, The Epsilon-Delta Definition of a Limit

Last revised: 2021-06-15 07:33

Objectives

- Interpret an inequality of the form $0 < |x - a| < c$ as a statement about the distance between x and a.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist. *(Moved from Lesson 2, §2.2)*
- Describe the idea behind the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

Inequalities representing distance

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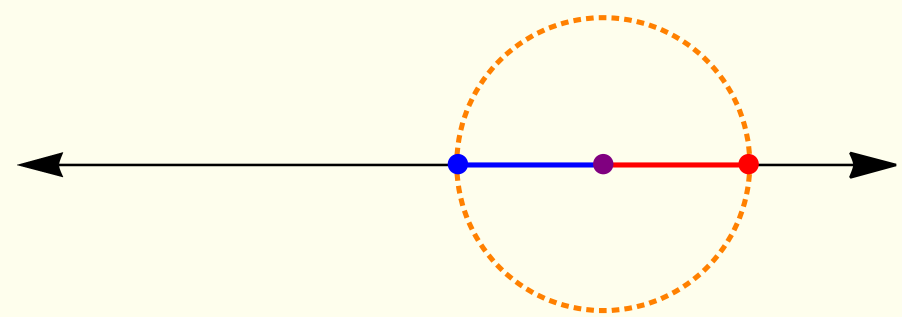


Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

Next, I'd mislead the students into incorrectly guessing the value of the limit of $\sin(1/x)$ as $x \to 0$, using a table of values.

Ex. 5. Guess the value of $\lim_{x\to 0} \sin \frac{1}{x}$.

Taking x closer and closer to 0, we find that:

x	$\sin \frac{1}{x}$
$\pm\frac{1}{\pi}$	0
$\pm\frac{1}{2\pi}$	0
$\pm\frac{1}{3\pi}$	0
$\pm\frac{1}{4\pi}$	0
$\pm\frac{1}{5\pi}$	0
$\pm\frac{1}{10\pi}$	0
$\pm\frac{1}{100\pi}$	0

Guess: $\lim_{x\to 0} \sin \frac{1}{x} = 0$

This time our guess is wrong.

Can you explain why by looking at the graph of $\sin \frac{1}{x}$?

Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

Then I'd say...

"This example shows that using a table of values to find a limit may mislead us."

Ex. 5. Guess the value of $\lim_{x\to 0} \sin \frac{1}{x}$.

Taking x closer and closer to 0, we find that:

x	$\sin \frac{1}{x}$
$\pm\frac{1}{\pi}$	0
$\pm\frac{1}{2\pi}$	0
$\pm\frac{1}{3\pi}$	0
$\pm\frac{1}{4\pi}$	0
$\pm\frac{1}{5\pi}$	0
$\pm\frac{1}{10\pi}$	0
$\pm\frac{1}{100\pi}$	0

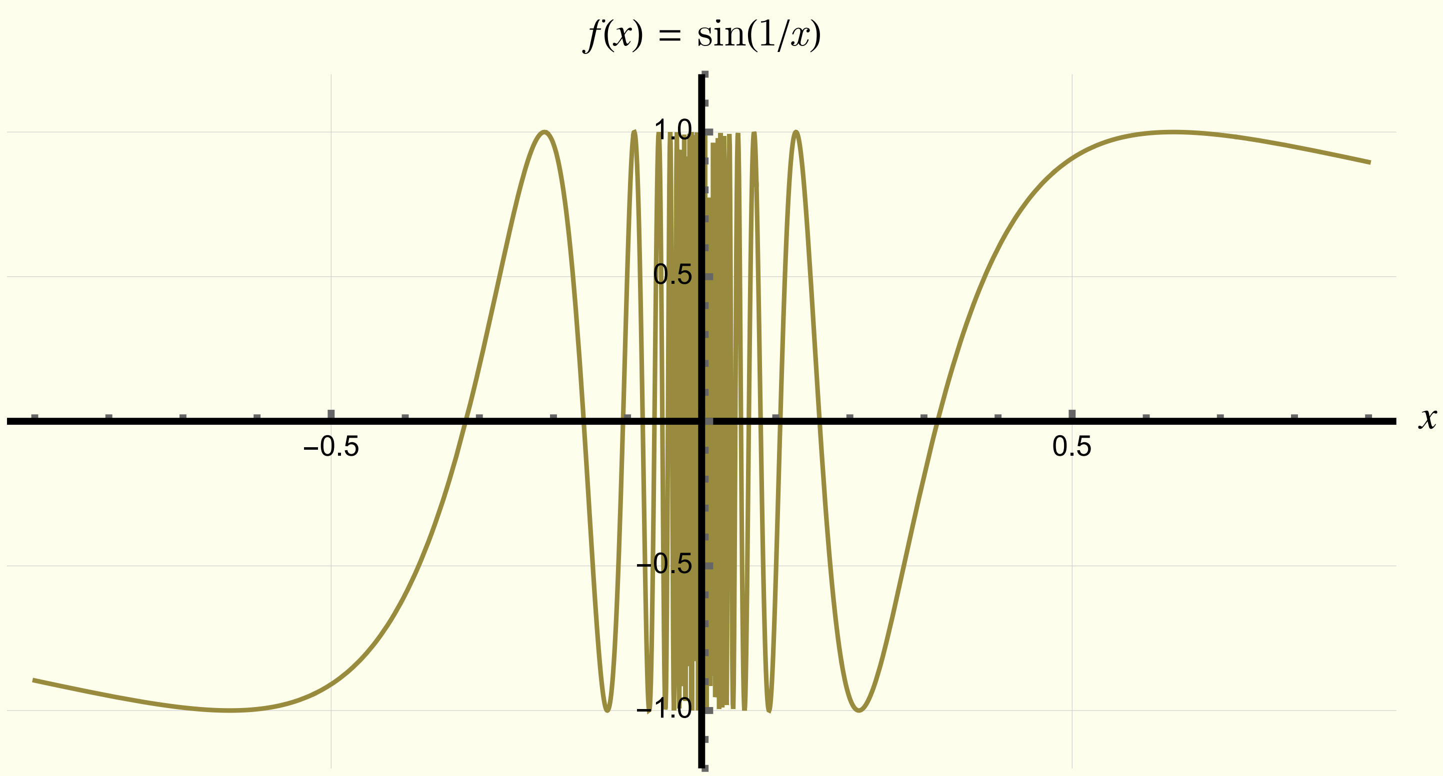
Guess: $\lim_{x\to 0} \sin \frac{1}{x} = 0$

This time our guess is wrong.

Can you explain why by looking at the graph of $\sin \frac{1}{x}$?

3. The Limit of $\sin(1/x)$ as $x \to 0$

The Workbook walks the student through what I would have done in person.



There's something seriously wrong with our "informal" definition of a limit—it misleads us into giving an incorrect answer. *The limit of* $\sin\frac{1}{x}$ *as* $x \to 0$ *does not exist.*

In the next section of this Lesson, we will revise our informal definition of

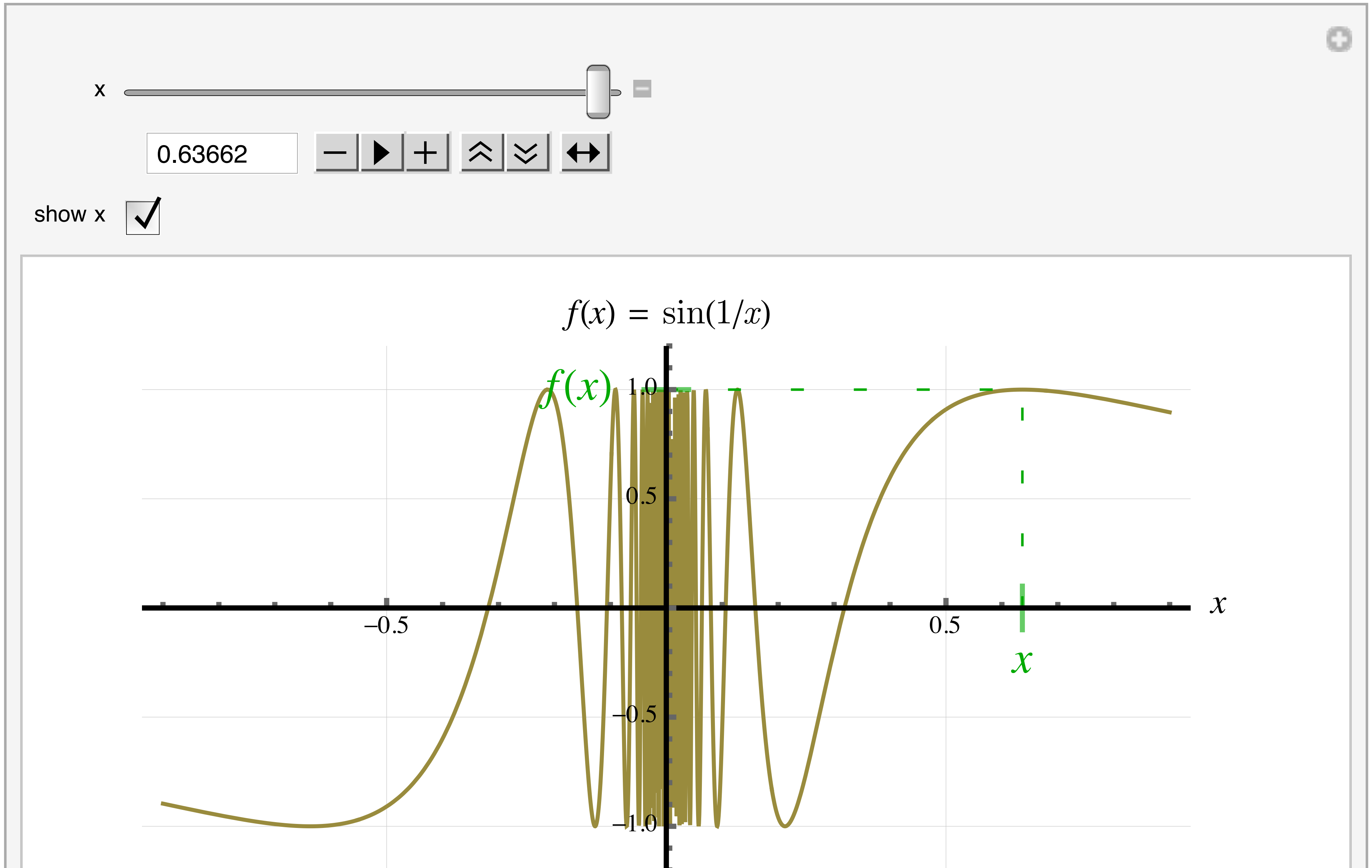
$$\lim_{x\to a} f(x)$$

and give a precise definition that is reliable in all cases.

Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

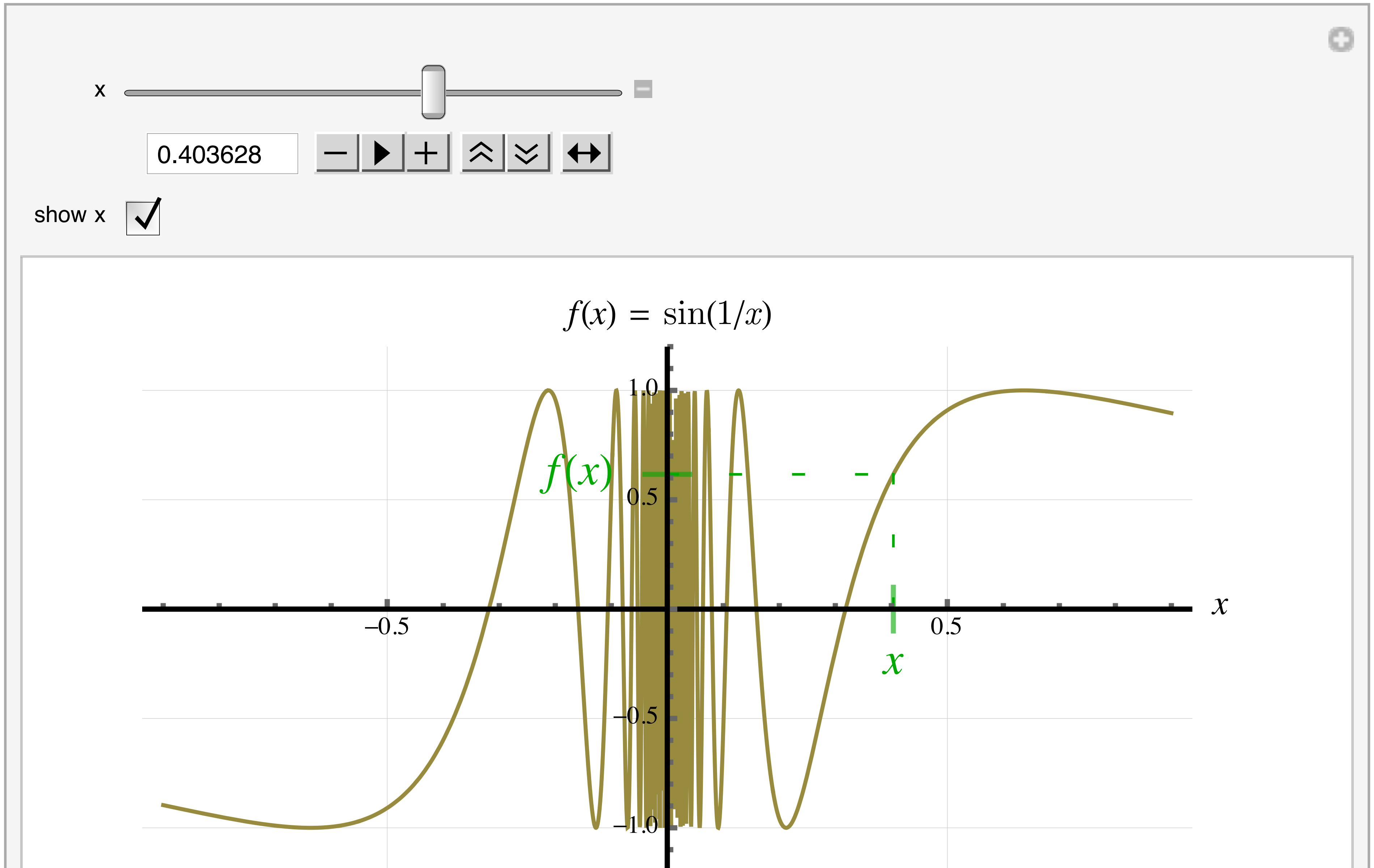
In person, I let this animation play at terrifically slow speed while asking the students what they think the limit is as $x \to 0$.



Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

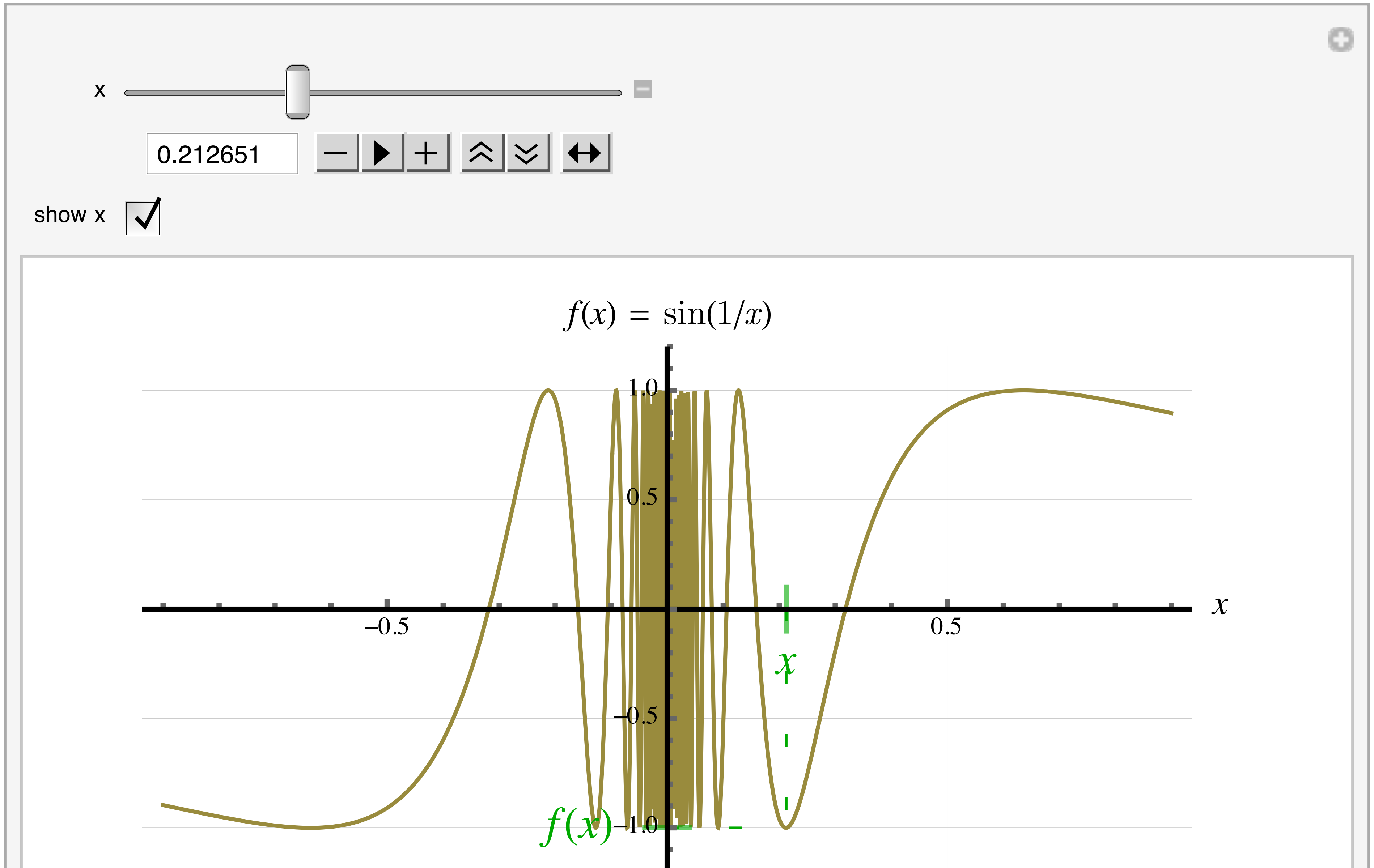
While x creeps along, we chat about the fact that the limit must be a *single* number, if it exists.



Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

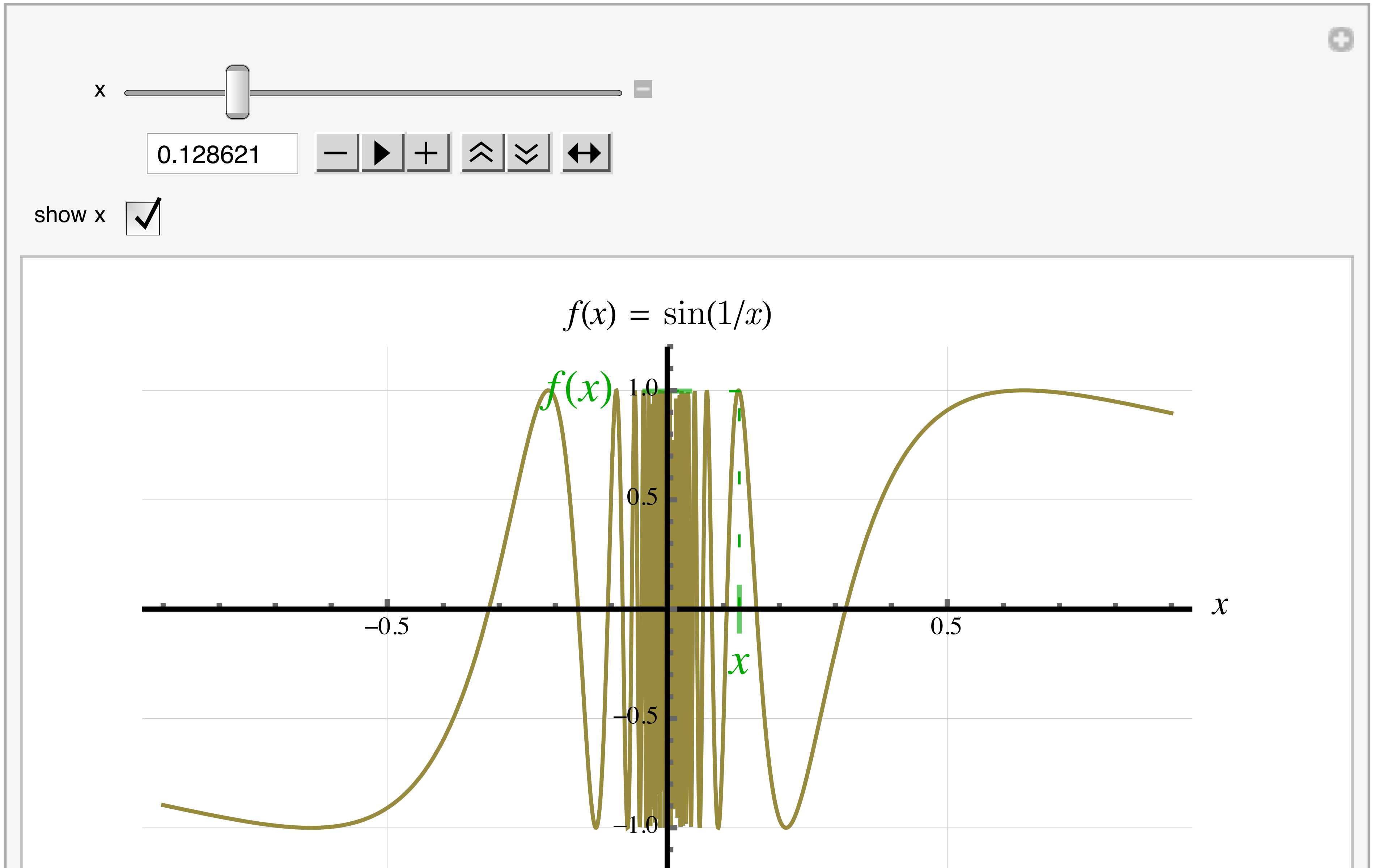
We discuss the fact that the function takes on the values 1 and -1 again and again as $x \to 0$.



Applets

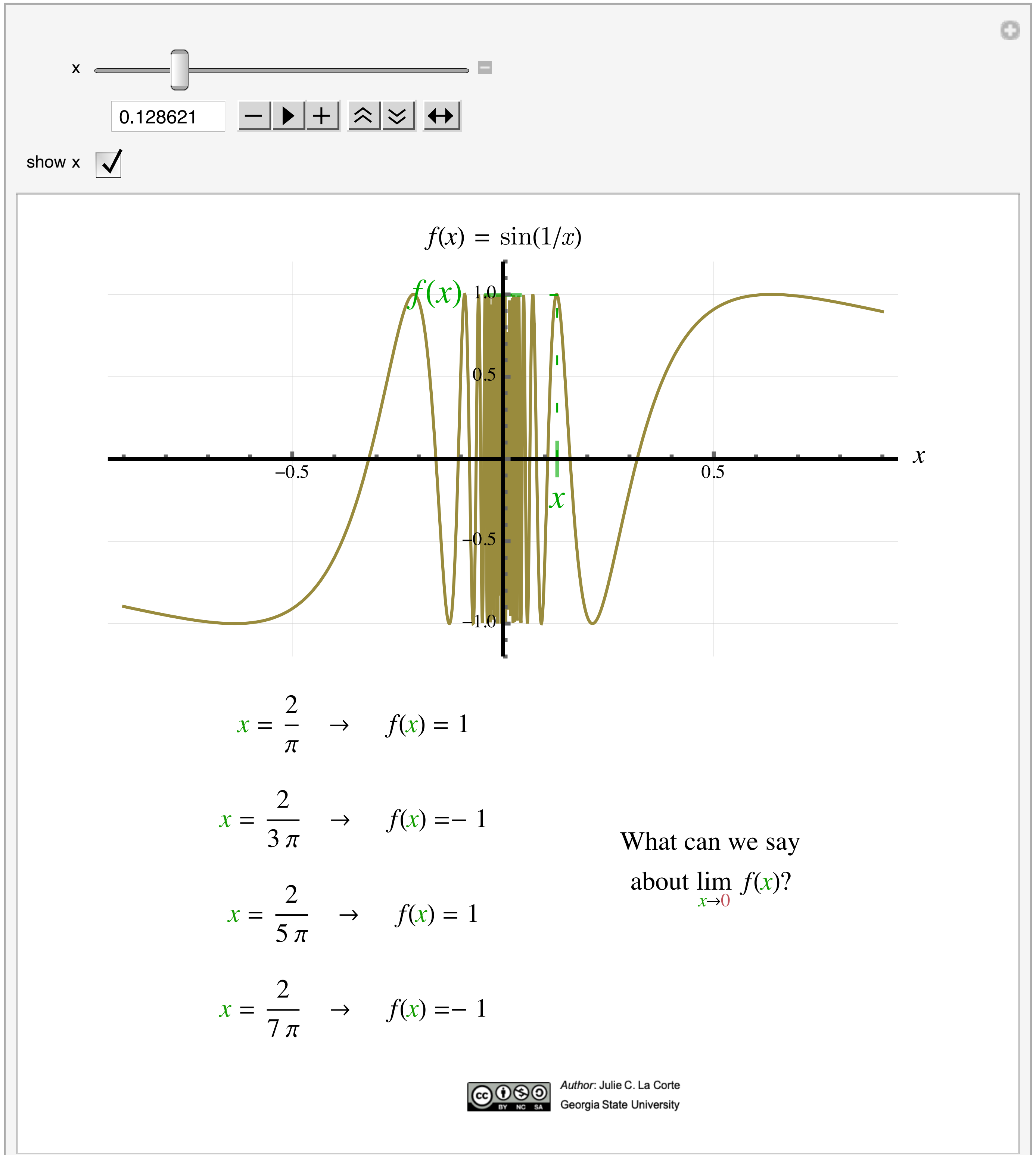
3. The Limit of $\sin(1/x)$ as $x \to 0$

Suspense builds in the classroom...



Applets

3. The Limit of $\sin(1/x)$ as $x \to 0$

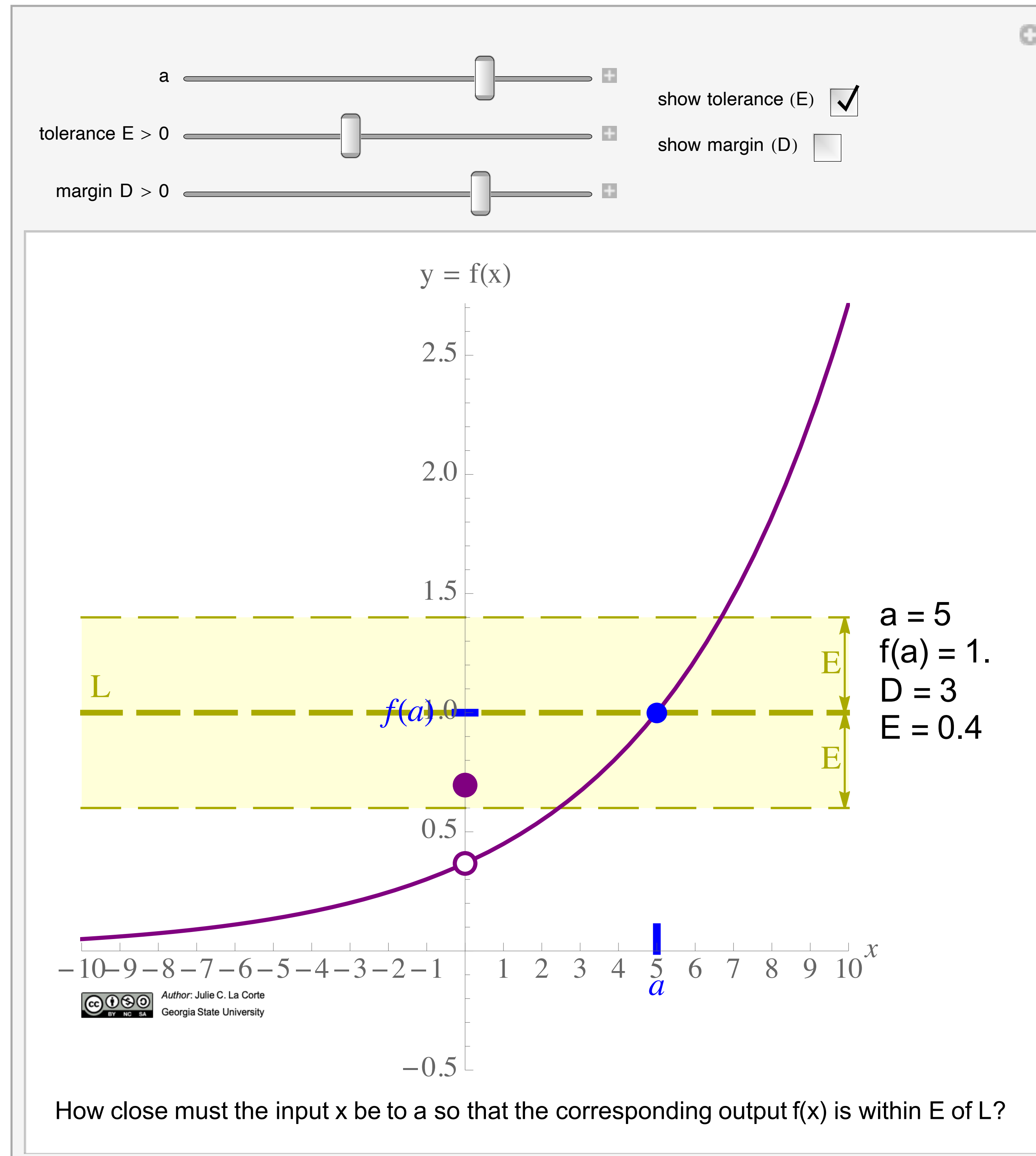


...but it soon becomes obvious that the value of $f(x)$ is not "homing in" on any single number as x approaches 0.

Applets

4. Formal Definition of the Limit of a Function

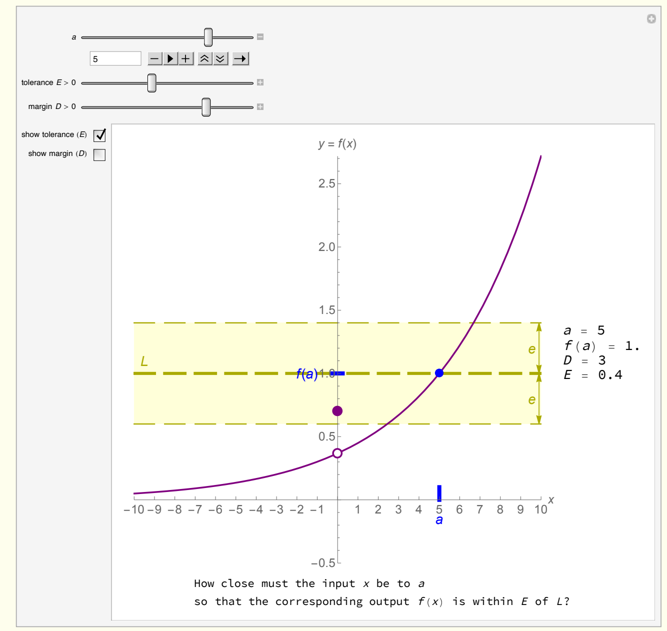
"Imagine an old-fashioned radio with a knob you turn to change the station. You don't have to tune the knob to exactly the right frequency. Within a certain tolerance will be close enough to make the radio station come in clearly."



4. Formal Definition of the Limit of a Function

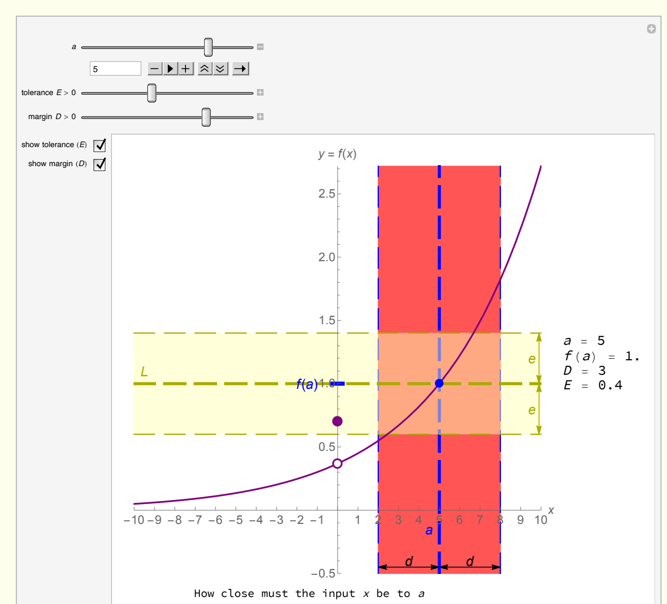
"Imagine an old-fashioned radio with a knob you turn to change the station. You don't have to tune the knob to exactly the right frequency. Within a certain tolerance will be close enough to make the radio station come in clearly."

You might say, well, how near do you want it? *(Set $a = 5$ in the applet.)* Let's say I want the output to be within *four tenths* of $f(a) = 1$. *(Set $E = .4$ in the applet.)*



Imagine an old-fashioned radio with a knob you turn to change the station. You don't have to tune the knob to exactly the right frequency. Within a certain tolerance will be close enough to make the radio station come in clearly.

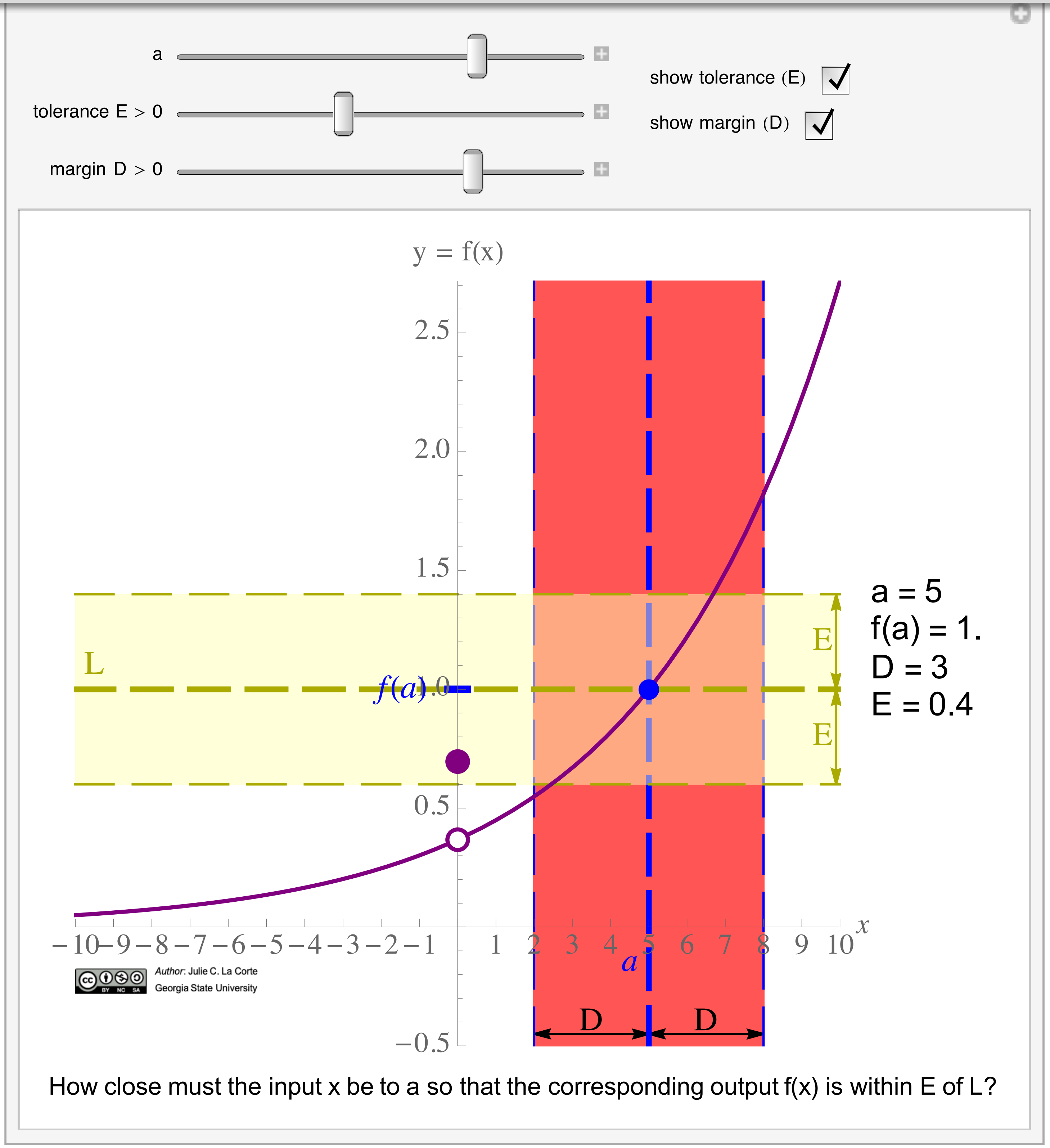
So how close to a do our x-values have to be to give us output values that are all within the tolerance shown? Is it enough to be within 3 units? *(Set $D = 3$ in the applet.)*



Applets

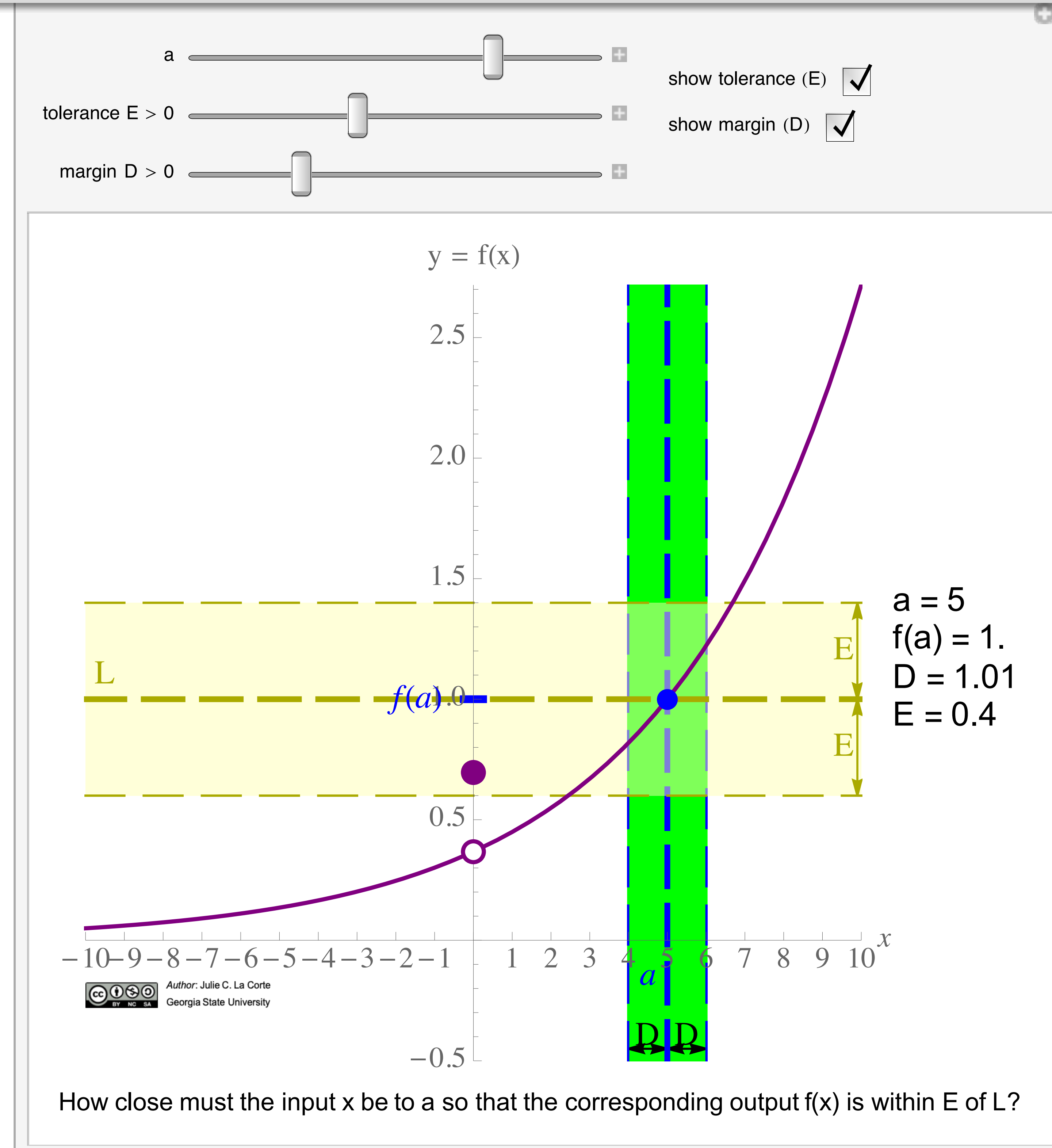
4. Formal Definition of the Limit of a Function

In non-pandemic times, I'd ask a student to work the controls while I guide class discussion.



4. Formal Definition of the Limit of a Function

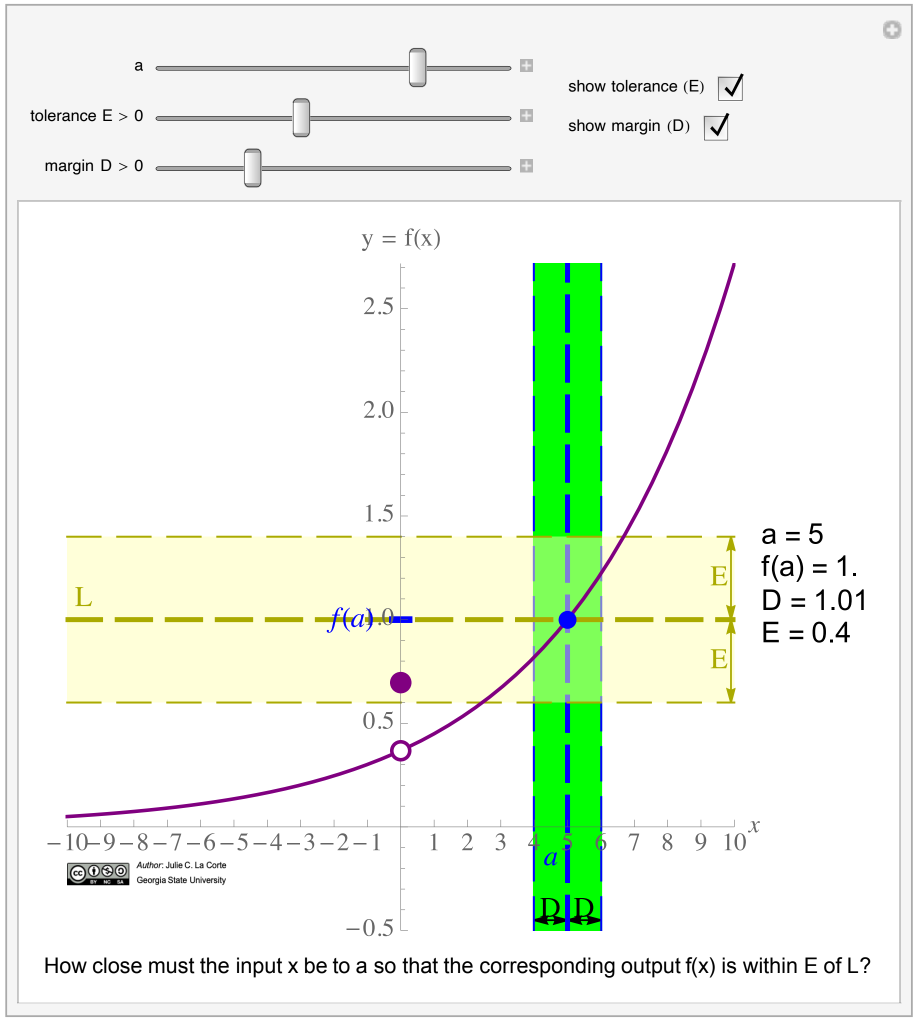
The change in color from red to green indicates that a suitable δ has been found for the given ε.



Applets

4. Formal Definition of the Limit of a Function

The text in the Workbook attempts to capture the feel of an in-person class.



4. Formal Definition of the Limit of a Function

That is, can I always make the margin stripe so small that the portion of the graph it contains is entirely contained in the yellow stripe—*no matter how narrow I make the yellow stripe*?

Yes. And this is the idea of a limit.

Formal definition. The statement

$$\lim_{x \to a} f(x) = L$$

(in words: "the **limit of** $f(x)$ **as** x **approaches** a is L") means that, given any tolerance $E > 0$, there exists some margin $D > 0$ such that

$$\left|f(x) - L\right| < E$$

whenever

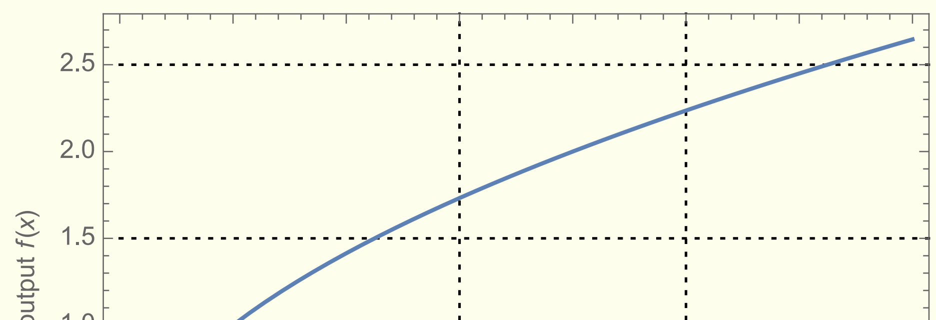
$$0 < |x - a| < D.$$

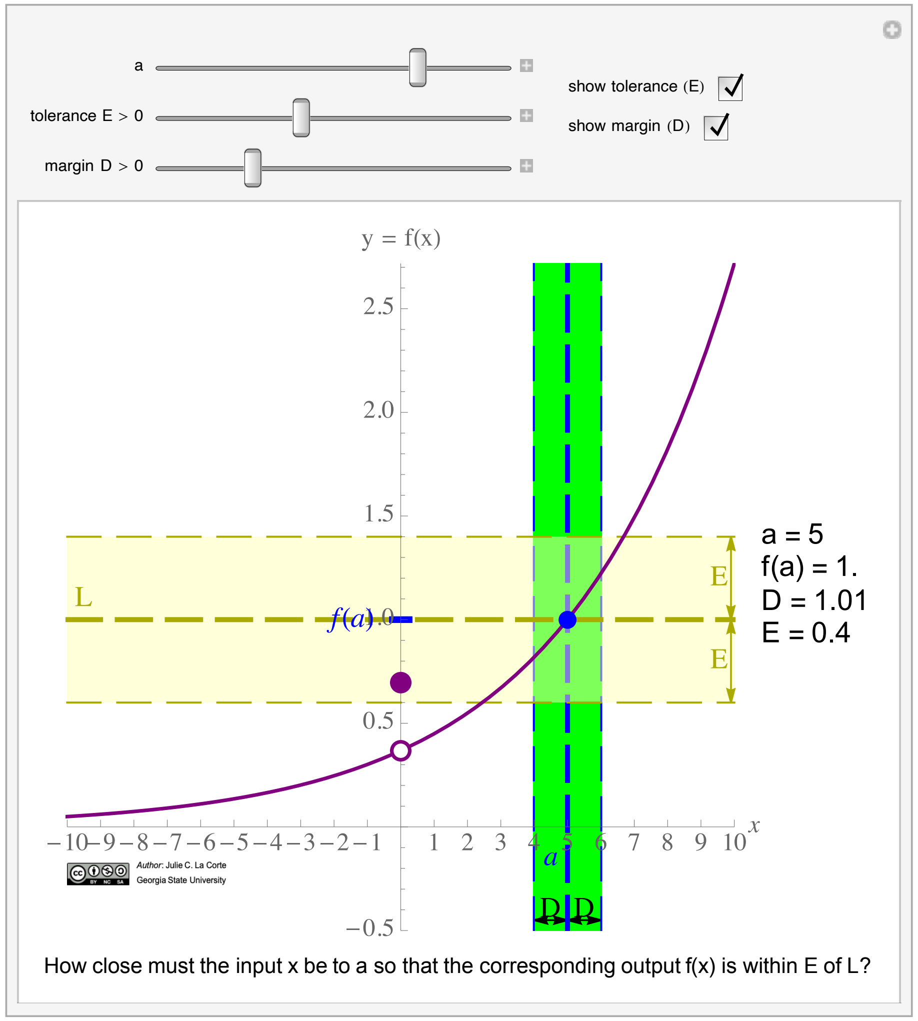
☞ We don't care what happens when $x = a$.

Note: Most authors use the Greek letters δ and ε in the above definition rather than the Roman letters D and E. (Of course, the names of variables don't matter in mathematics!)

Ex. 7. Use the graph provided below to complete the statement:

$$|f(x) - 2| < \_\_\_\_ \text{ whenever } \_\_\_\_ < x < \_\_\_\_.$$



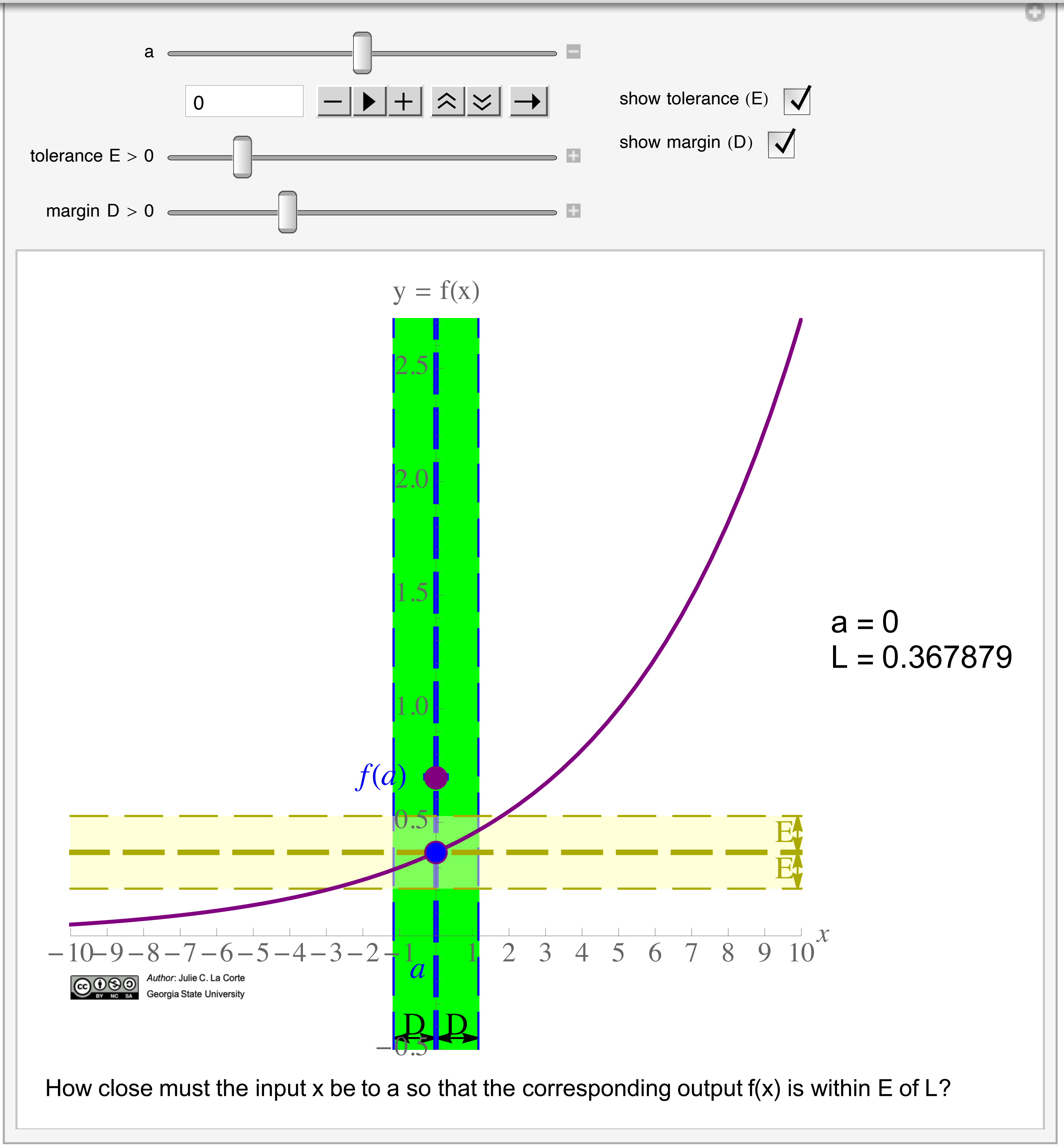


Applets

4. Formal Definition of the Limit of a Function

The function graphed in the applet has a removable discontinuity to facilitate discussion of the case

$$\lim_{x \to a} f(x) \neq f(a).$$



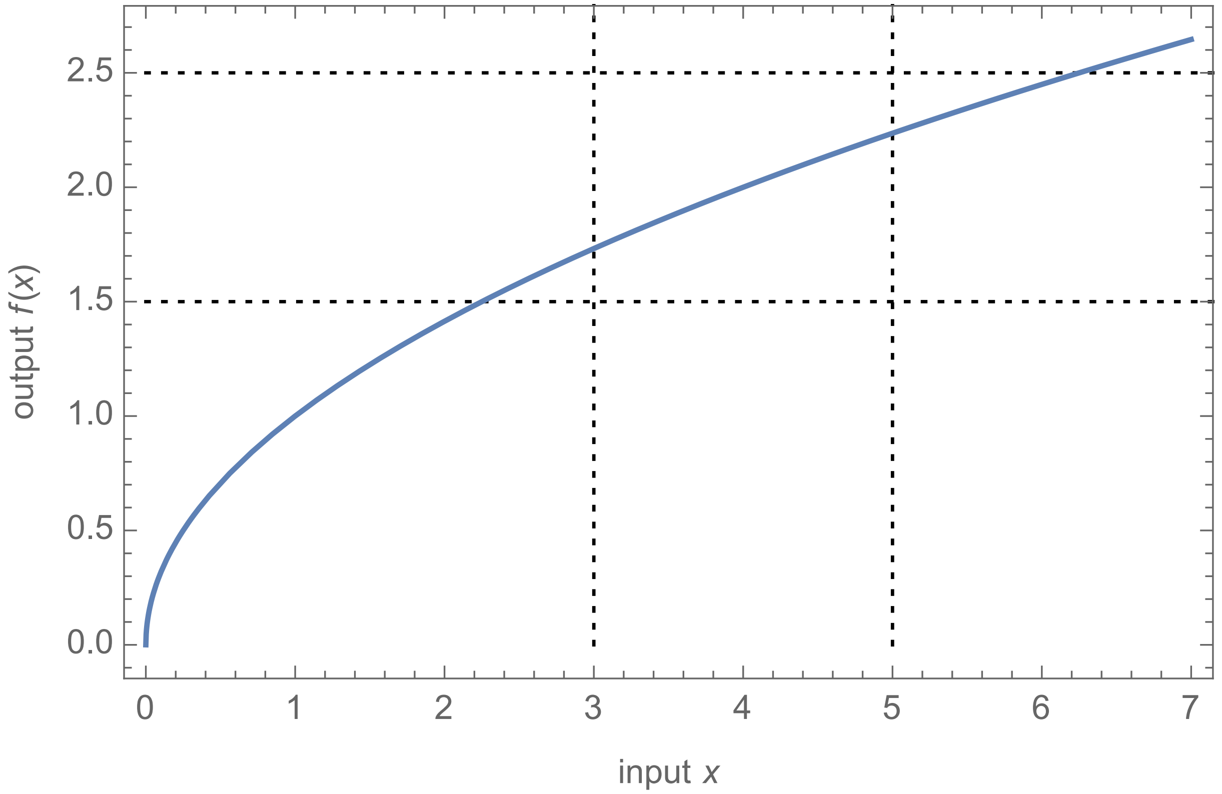
Applets

4. Formal Definition of the Limit of a Function

On an exam, *most students correctly answered a problem of this type*:

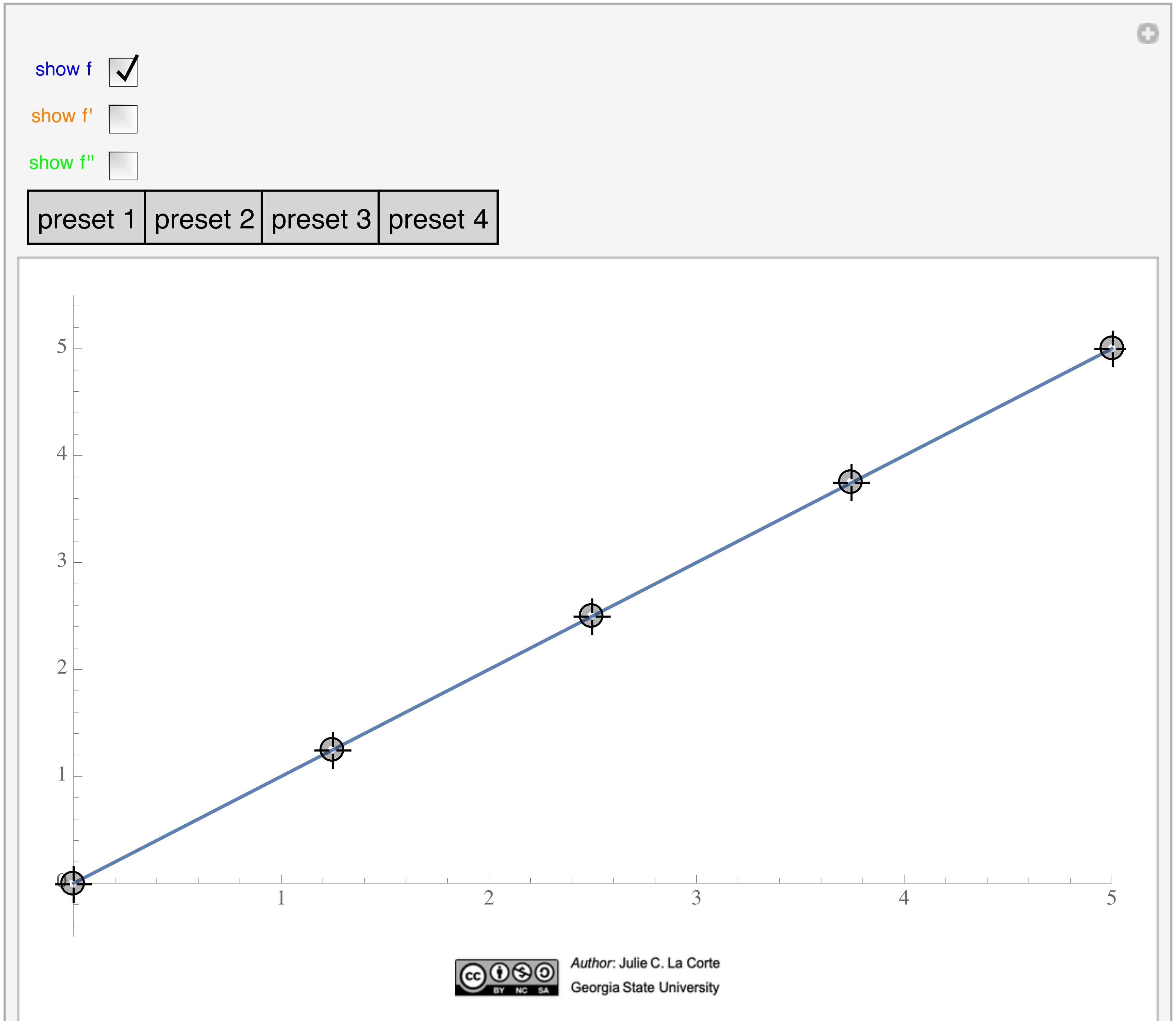
Ex. 7. Use the graph provided below to complete the statement:

$$|f(x) - 2| < \_\_\_\_ \text{ whenever } \_\_\_\_ < x < \_\_\_\_.$$



5. Derivative Sandbox

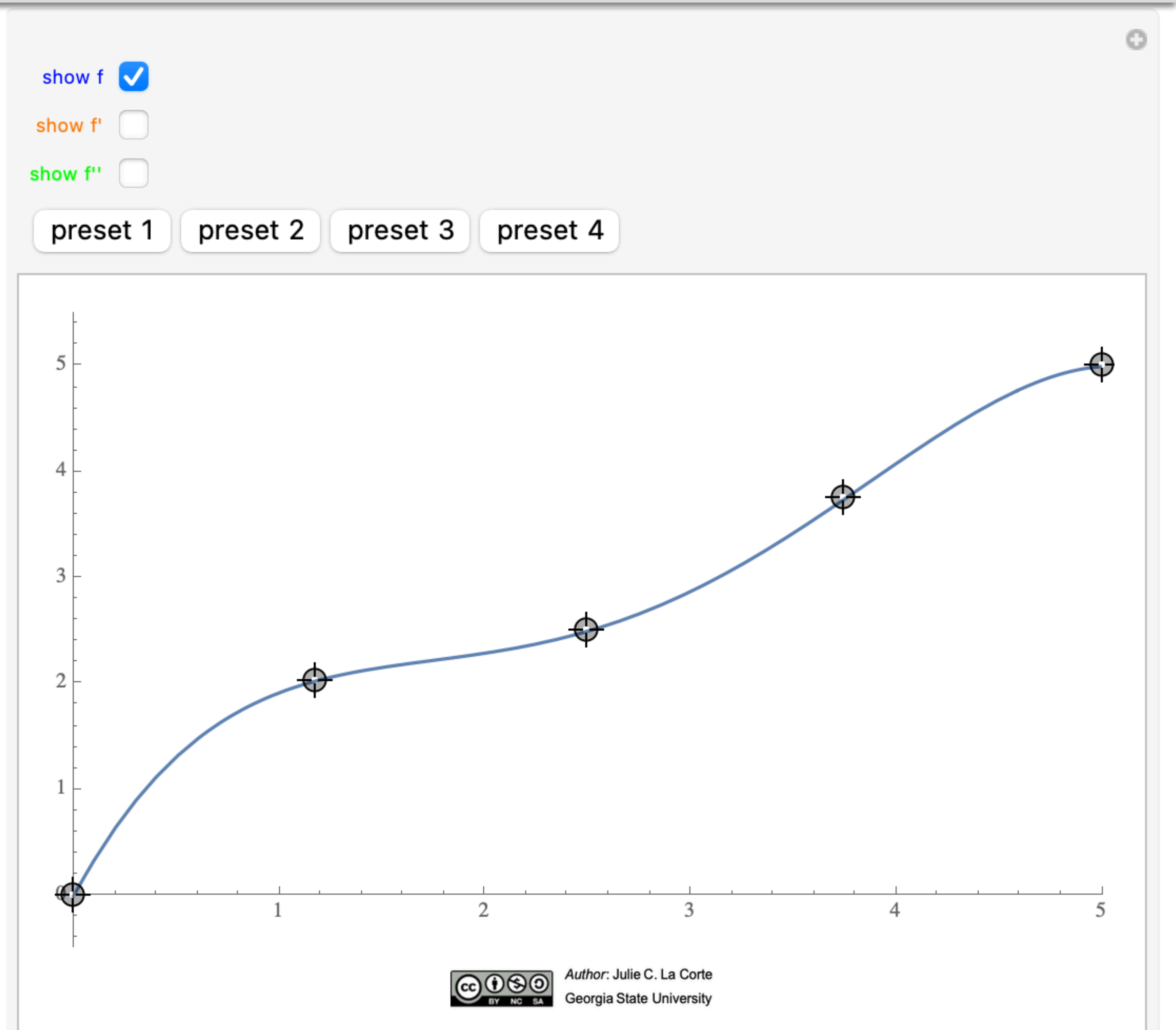
The "Derivative Sandbox" allows the student to freely experiment with the shape of a graph in order to see how the change impacts the first and second derivatives.



Applets

5. Derivative Sandbox

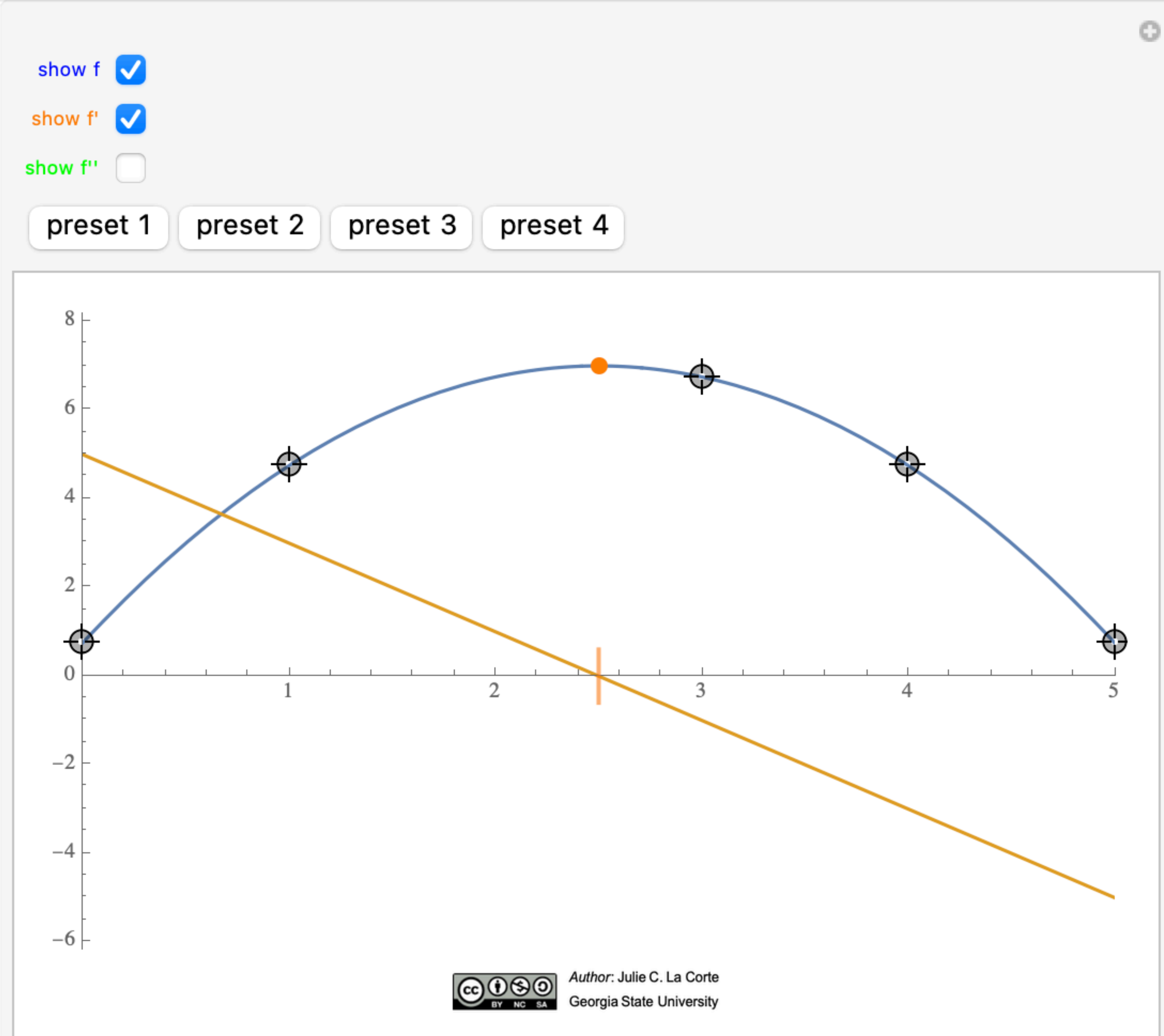
The crosshairs ⊕ can be dragged to bend the shape of the graph.



Applets

5. Derivative Sandbox

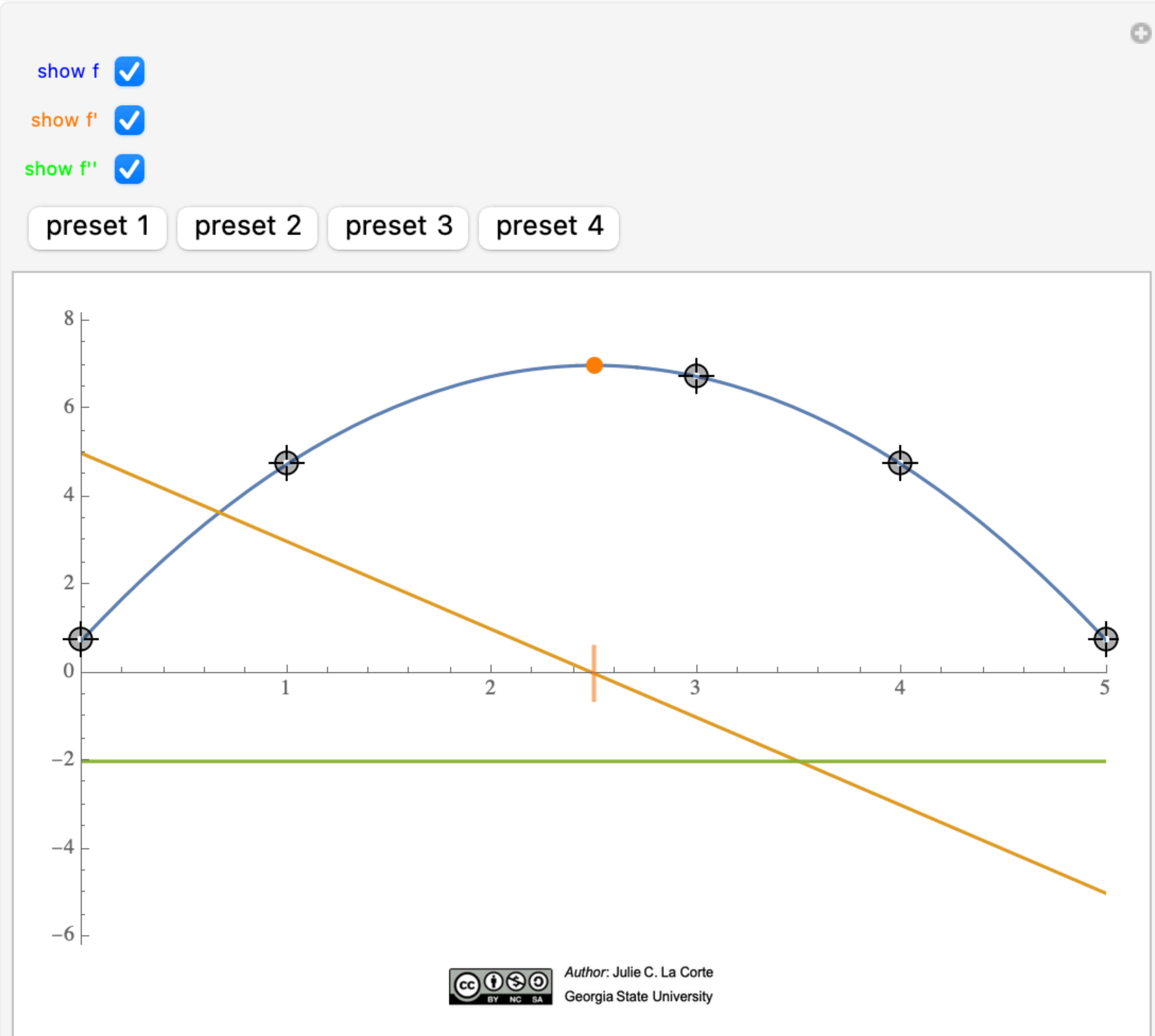
Four presets are provided (linear, quadratic, cubic, quartic).



Applets

5. Derivative Sandbox

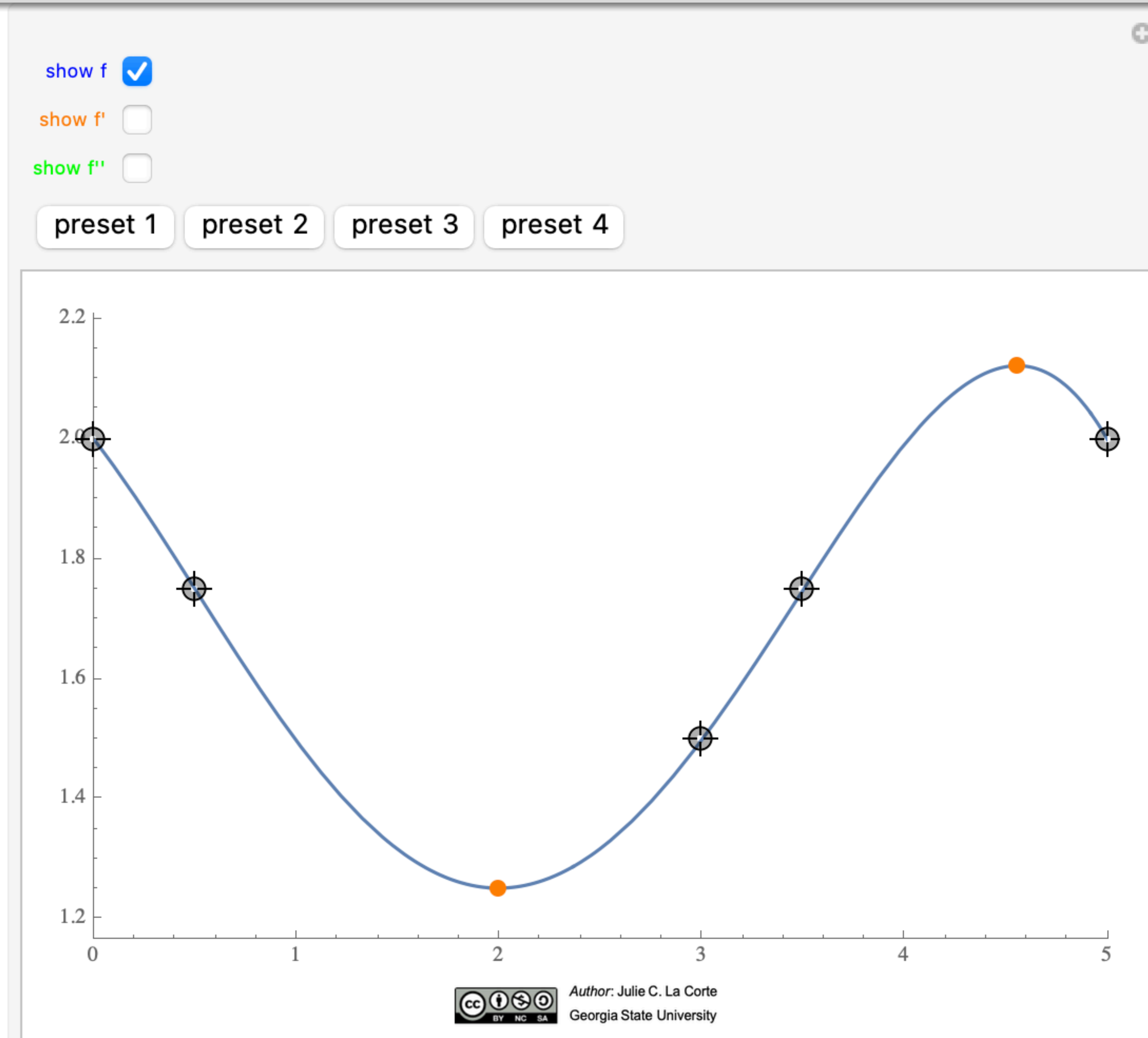
Even without prompting, students soon notice the orange dots which appear whenever a turning point is introduced.



Applets

5. Derivative Sandbox

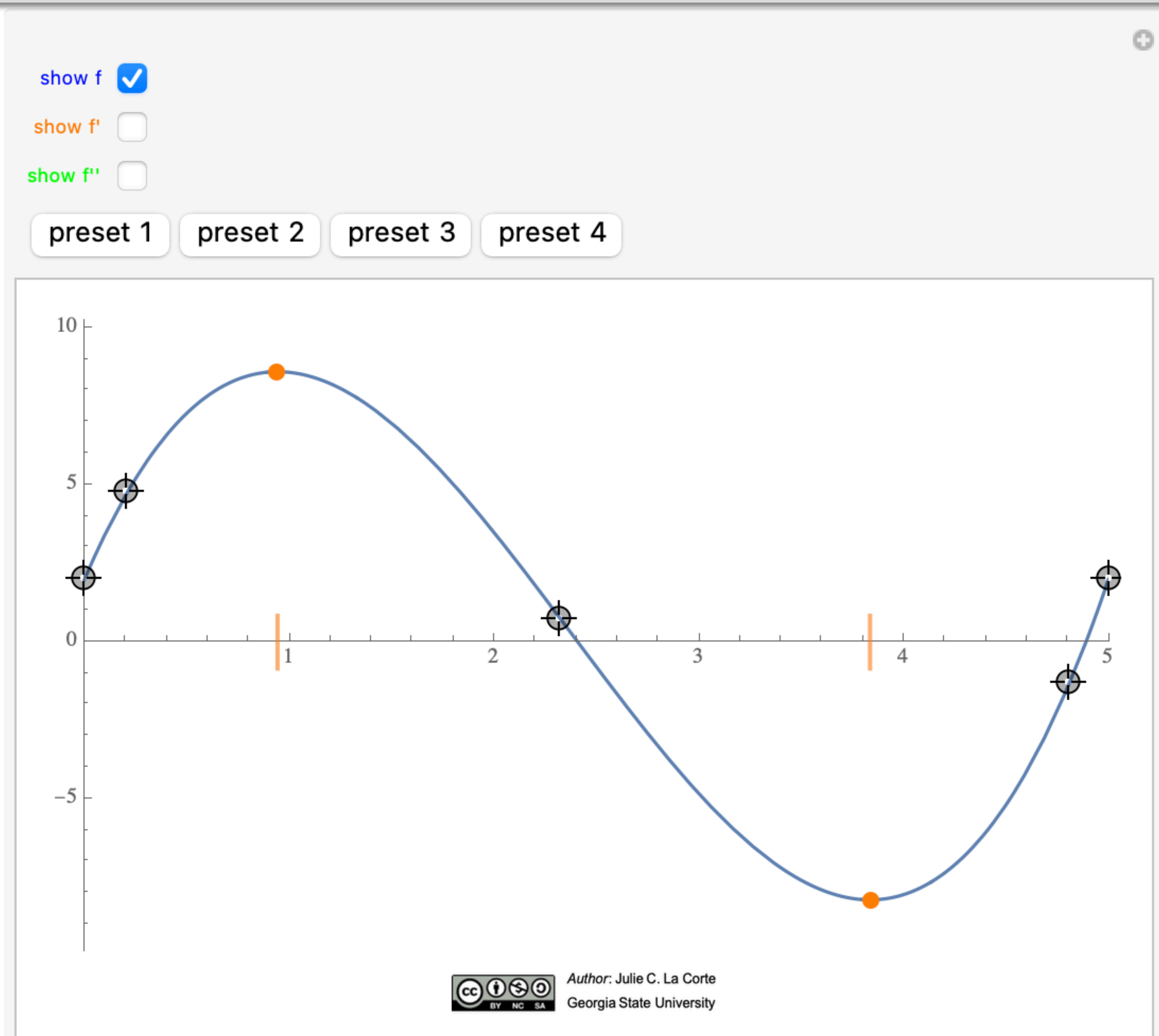
In class, the graph of f' can be hidden and revealed strategically by the instructor for discussion purposes.



Applets

5. Derivative Sandbox

But several students "discovered" relationships between the graphs of f and its first two derivatives just by playing around with the applet.



Applets

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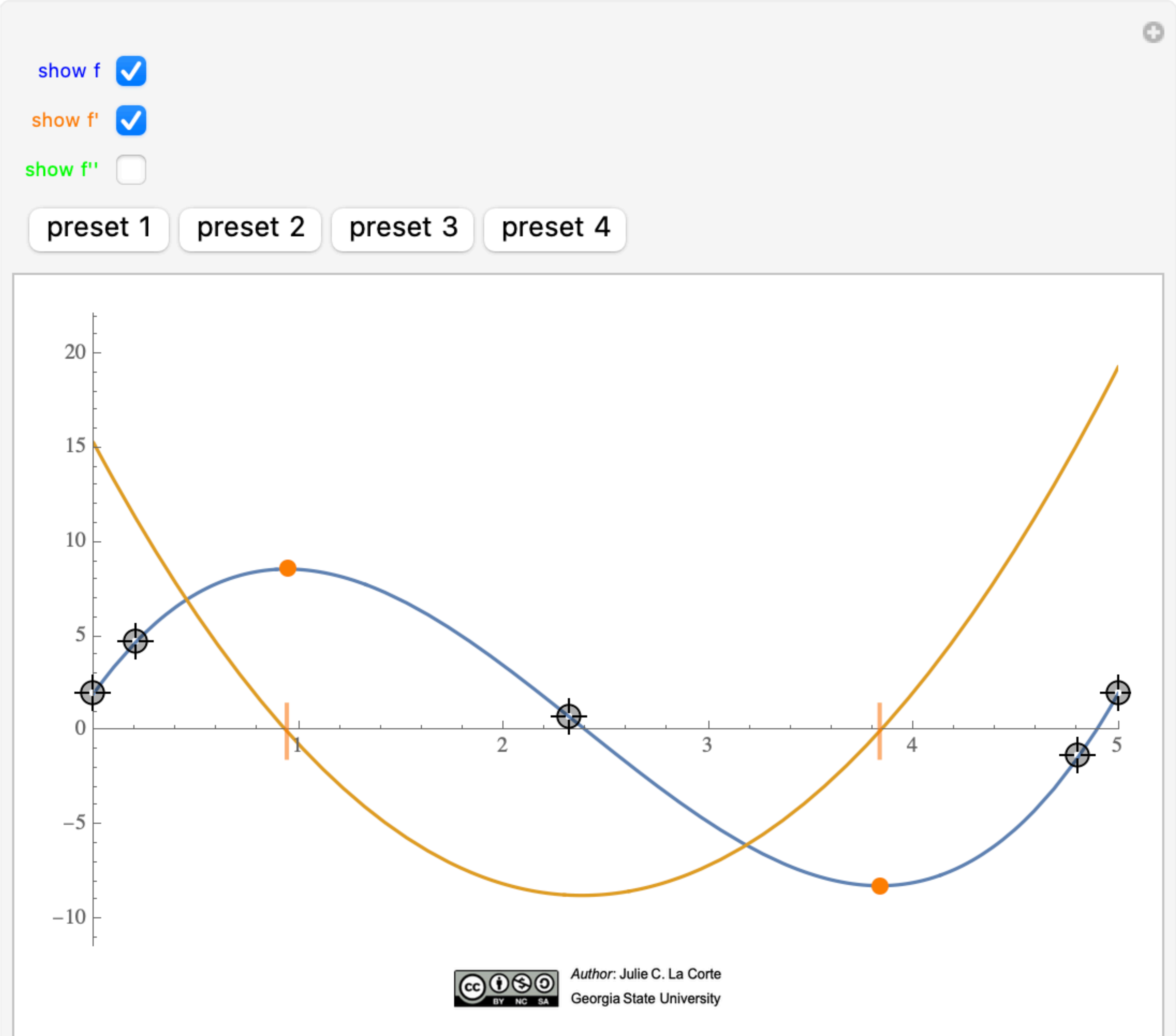

Applets

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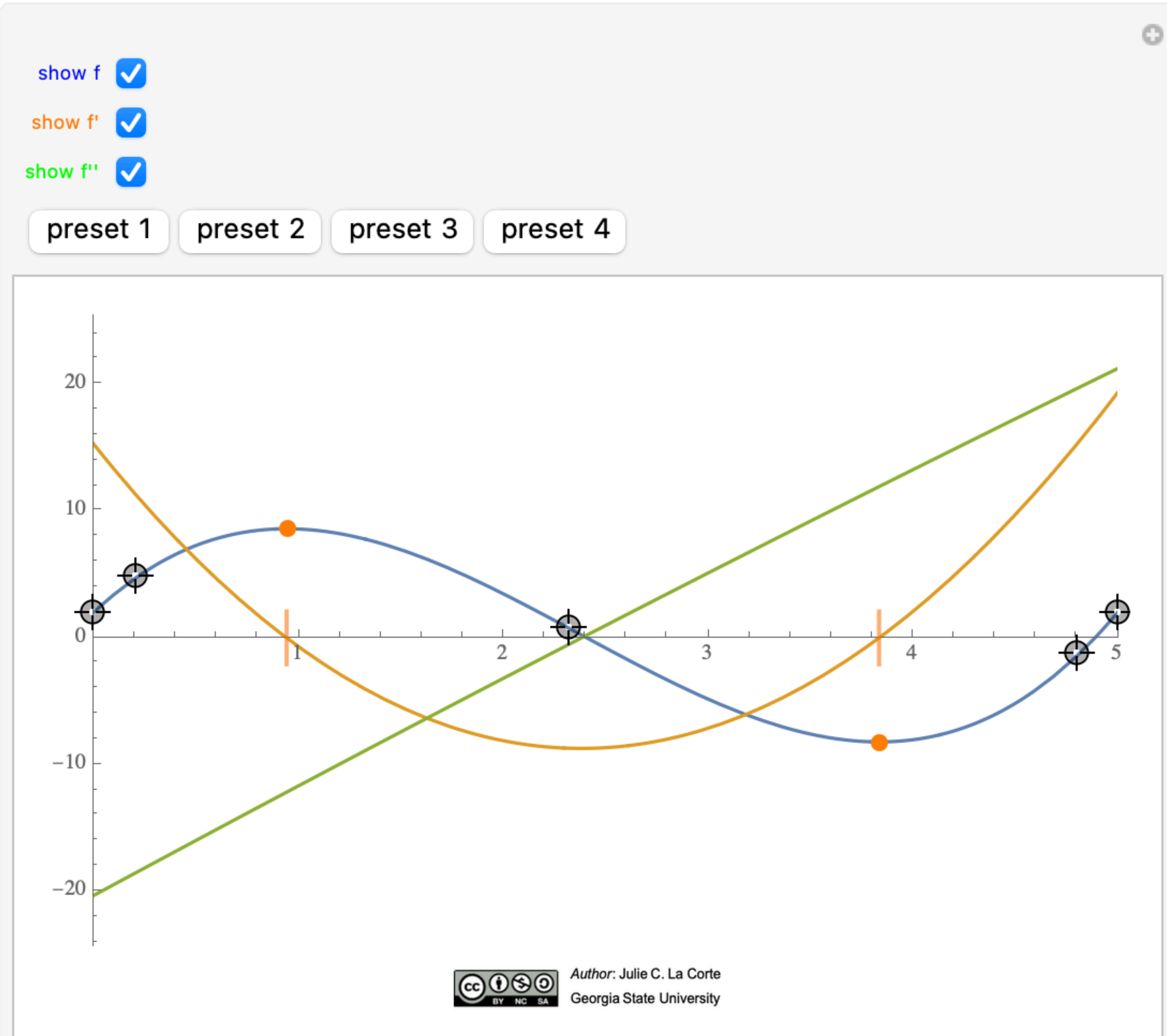


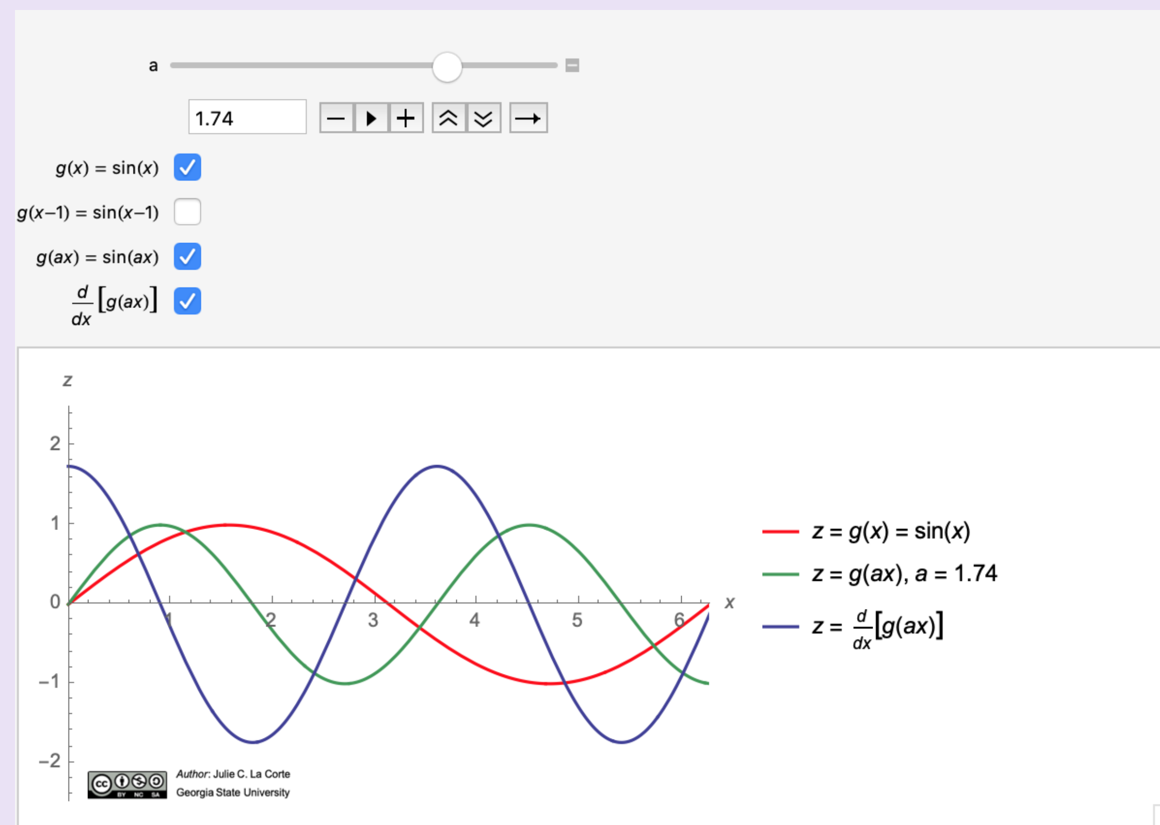
Applets

6. Introducing the Chain Rule

A standalone document lists the learning objectives I had in mind for each applet.

§3.4

- Motivation for Chain Rule

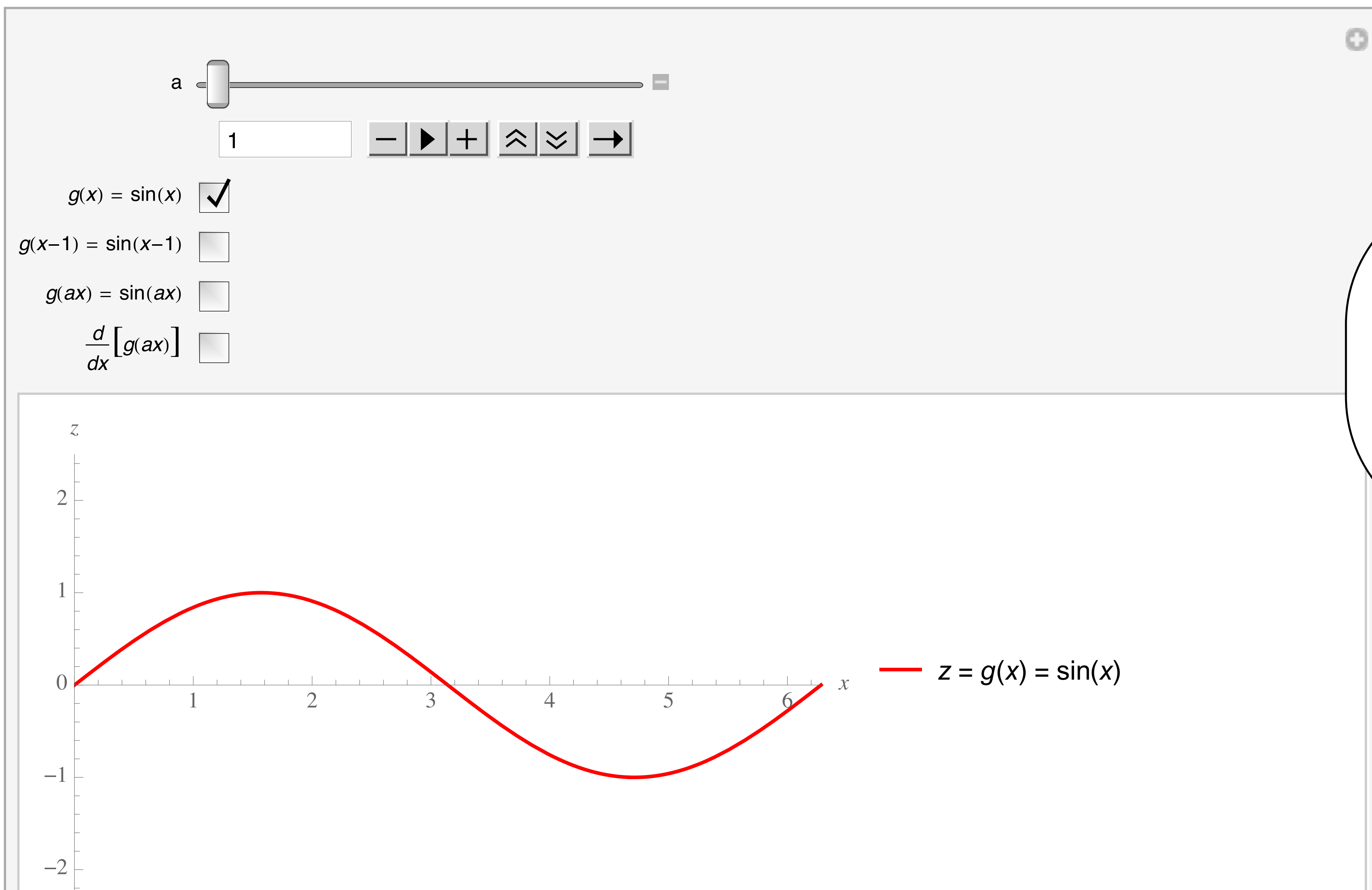


Objectives:

- Prior to formally presenting the Chain Rule, build intuition about the relationships between the derivatives of $g(x)$, $g(x-1)$, and $g(ax)$ $(x > 1)$ in general, taking $g(x) = \sin(x)$ for a concrete example.
- Prompt students to guess the derivative of $g(x-1)$ based on their intuitive understanding (e.g. of tangent lines).
- Illustrate how the graph of the derivative of $\sin(ax)$ changes amplitude when the value of a is varied.

Applets

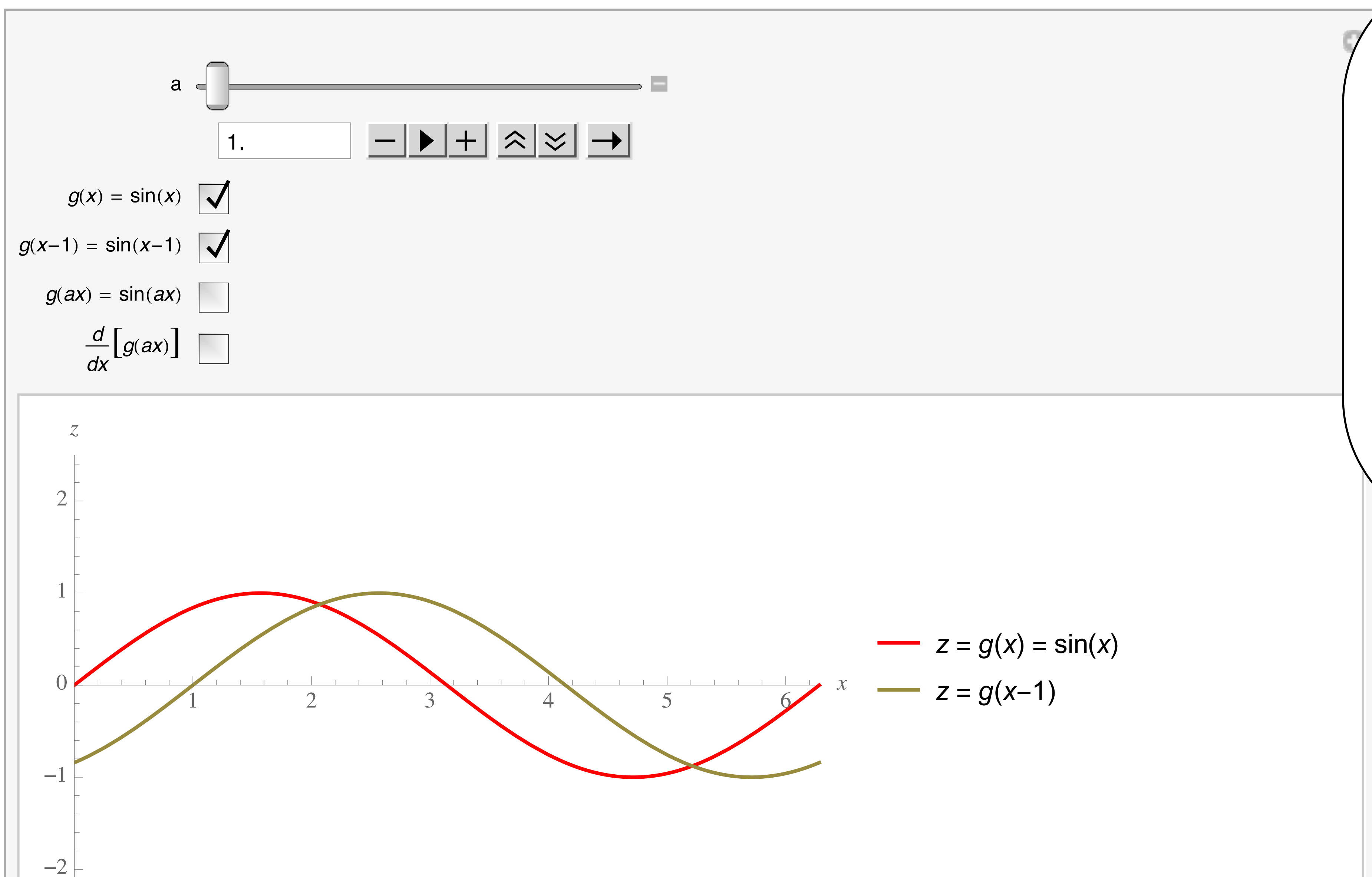
6. Introducing the Chain Rule



"What would you guess the derivative of $\sin(x-1)$ is?"

Applets

6. Introducing the Chain Rule



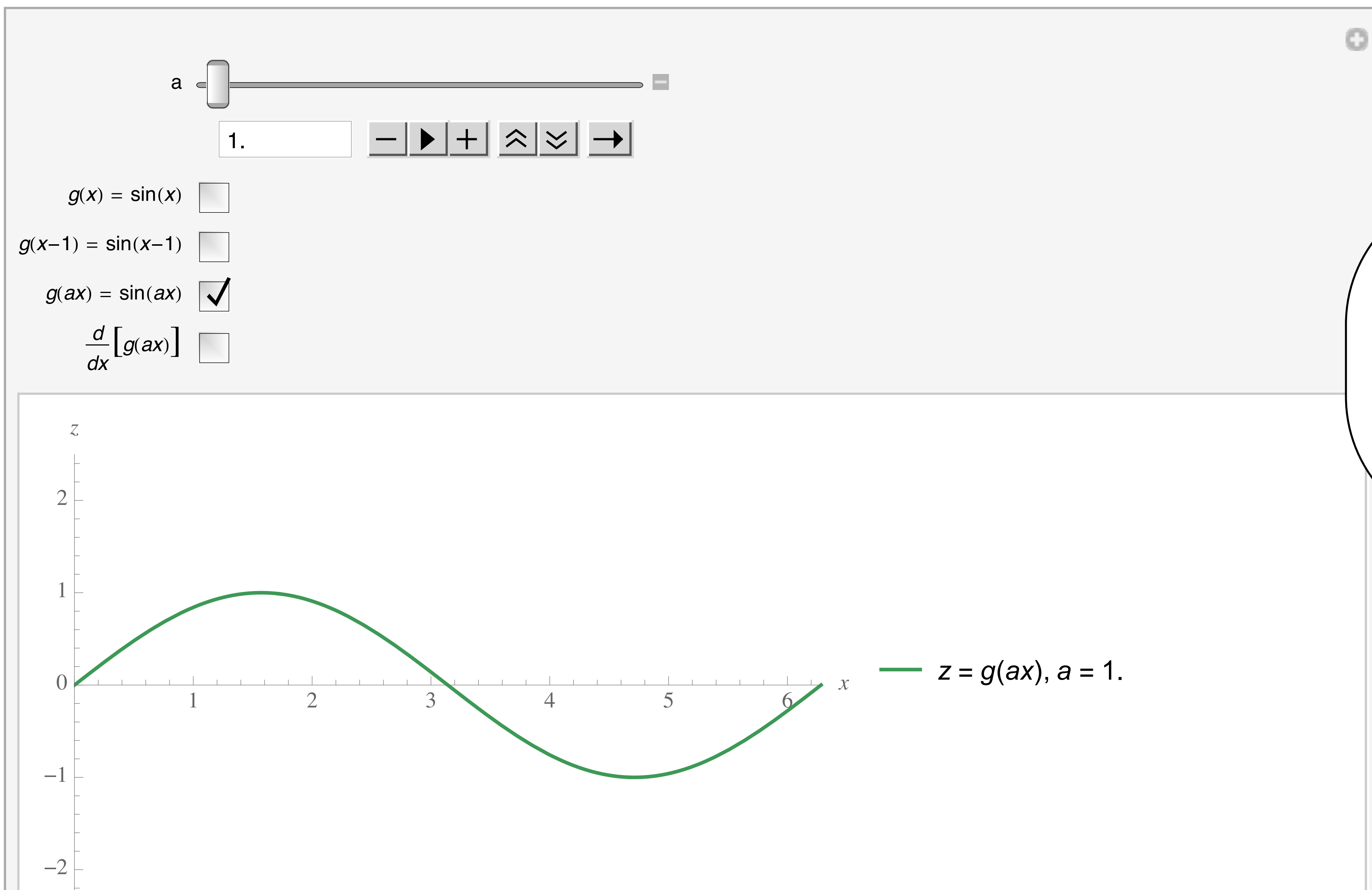
"The tangent line to $\sin(x)$ at $x = 0$ has the same slope as the tangent to the shifted version at $x = 1$."

Applets

6. Introducing the Chain Rule

The slider controls the frequency of the sinusoidal $\sin(ax)$.

As a is increased, students can see the tangent at 0 grow steeper.



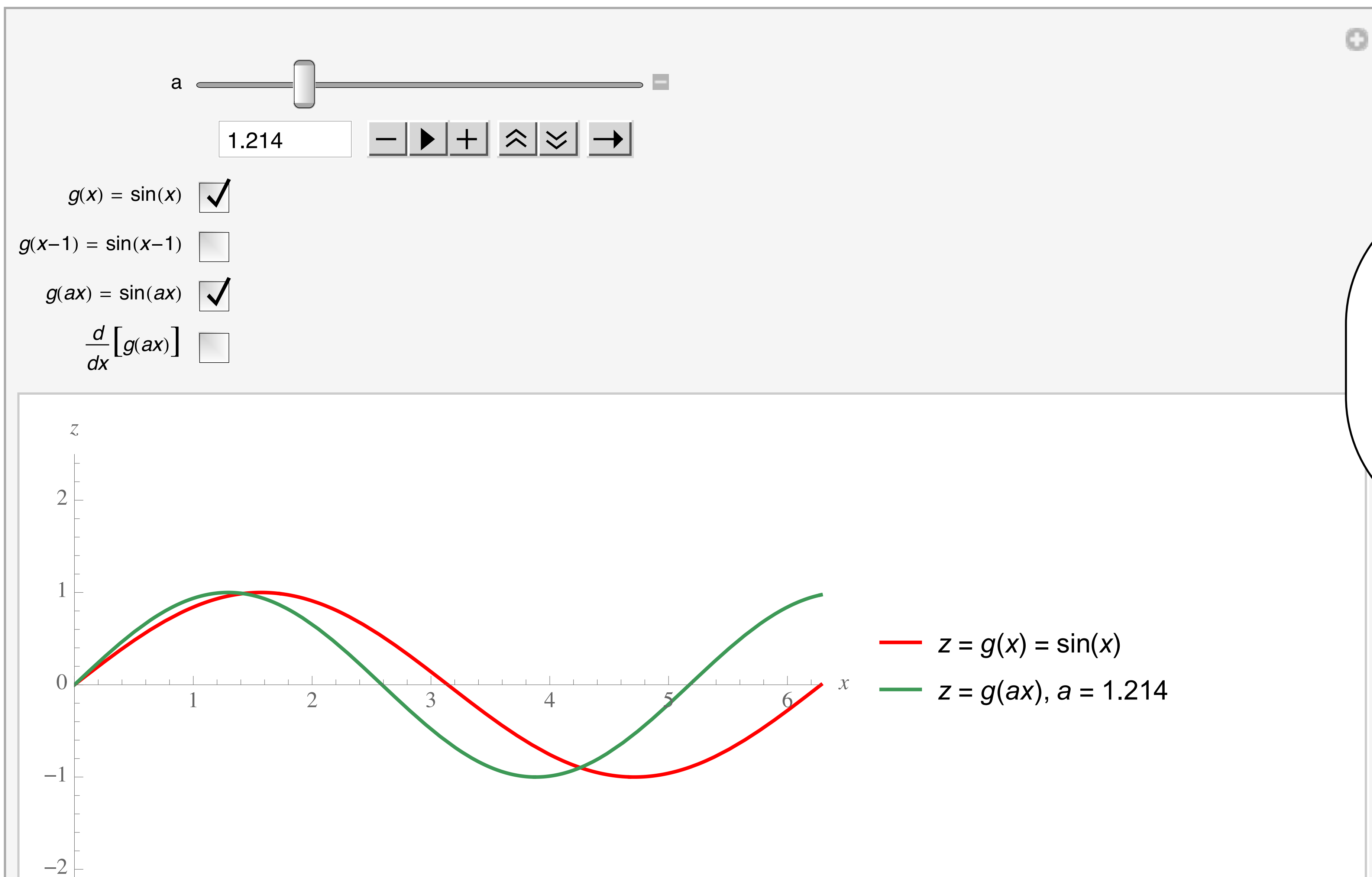
"What if we *speed up* $\sin(x)$?"

Applets

6. Introducing the Chain Rule

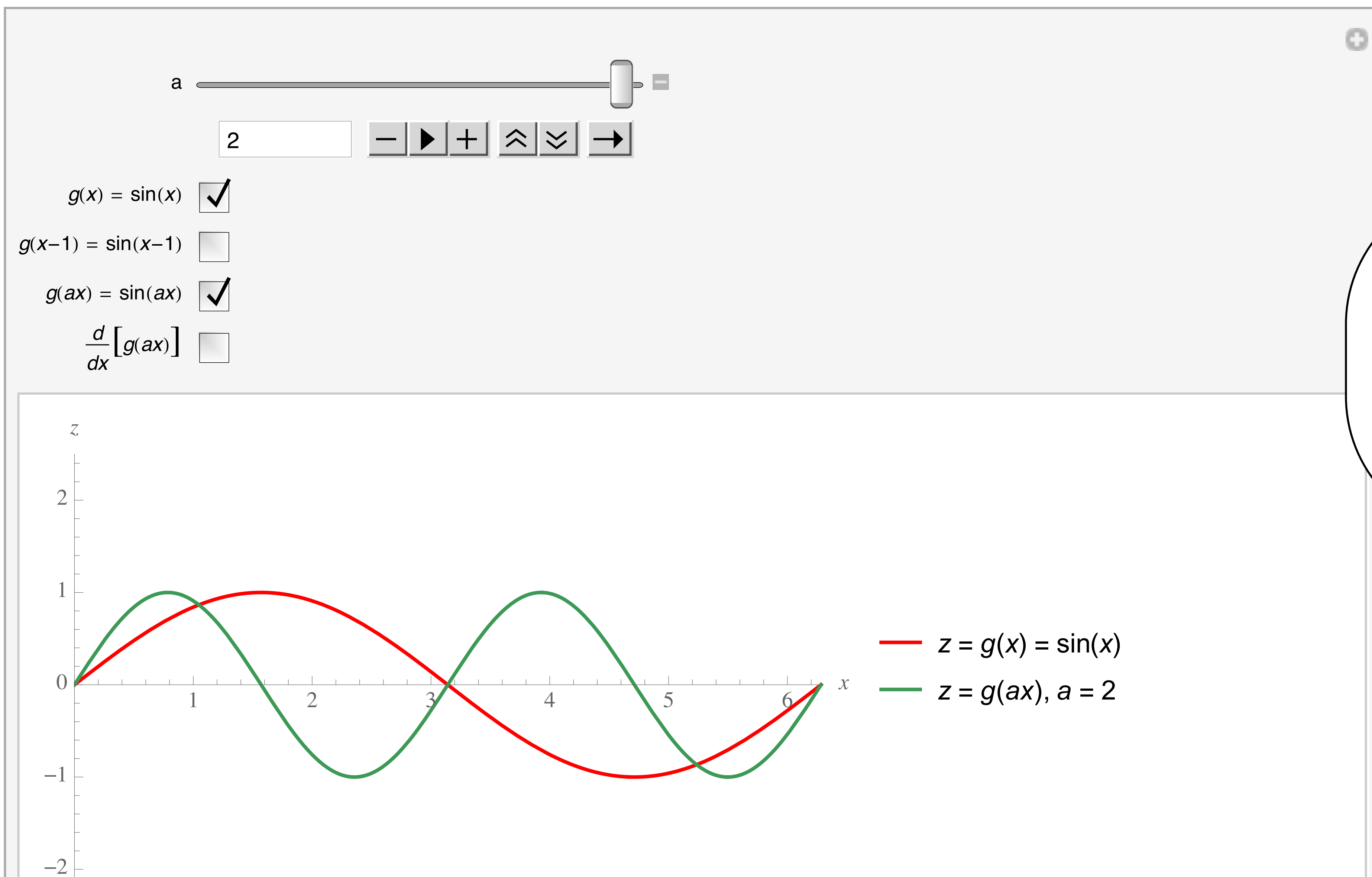
The slider controls the frequency of the sinusoidal $\sin(ax)$.

As a is increased, students can see the tangent at 0 grow steeper.



Applets

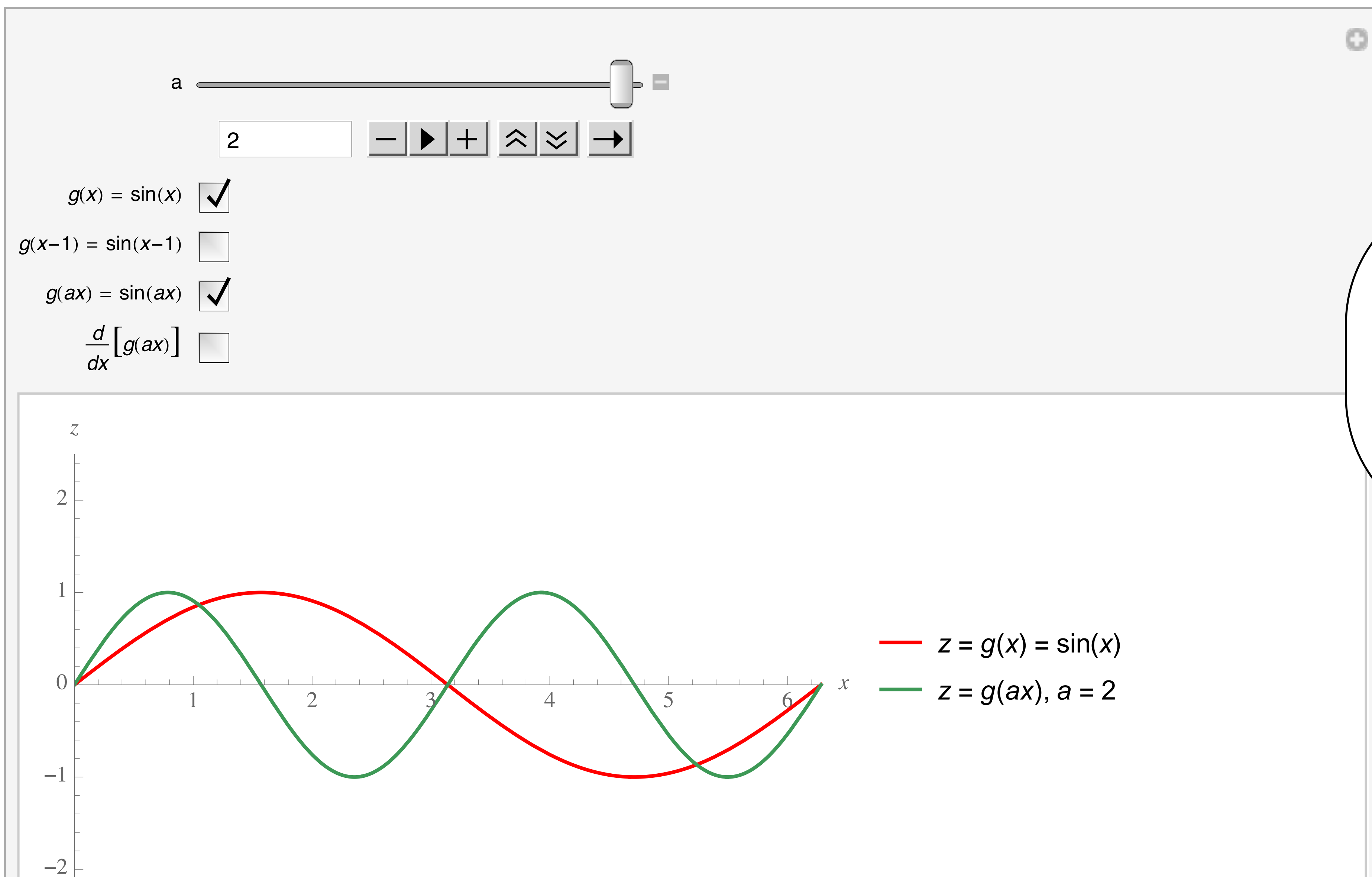
6. Introducing the Chain Rule



"…say, by doubling it, to get $\sin(2x)$."

Applets

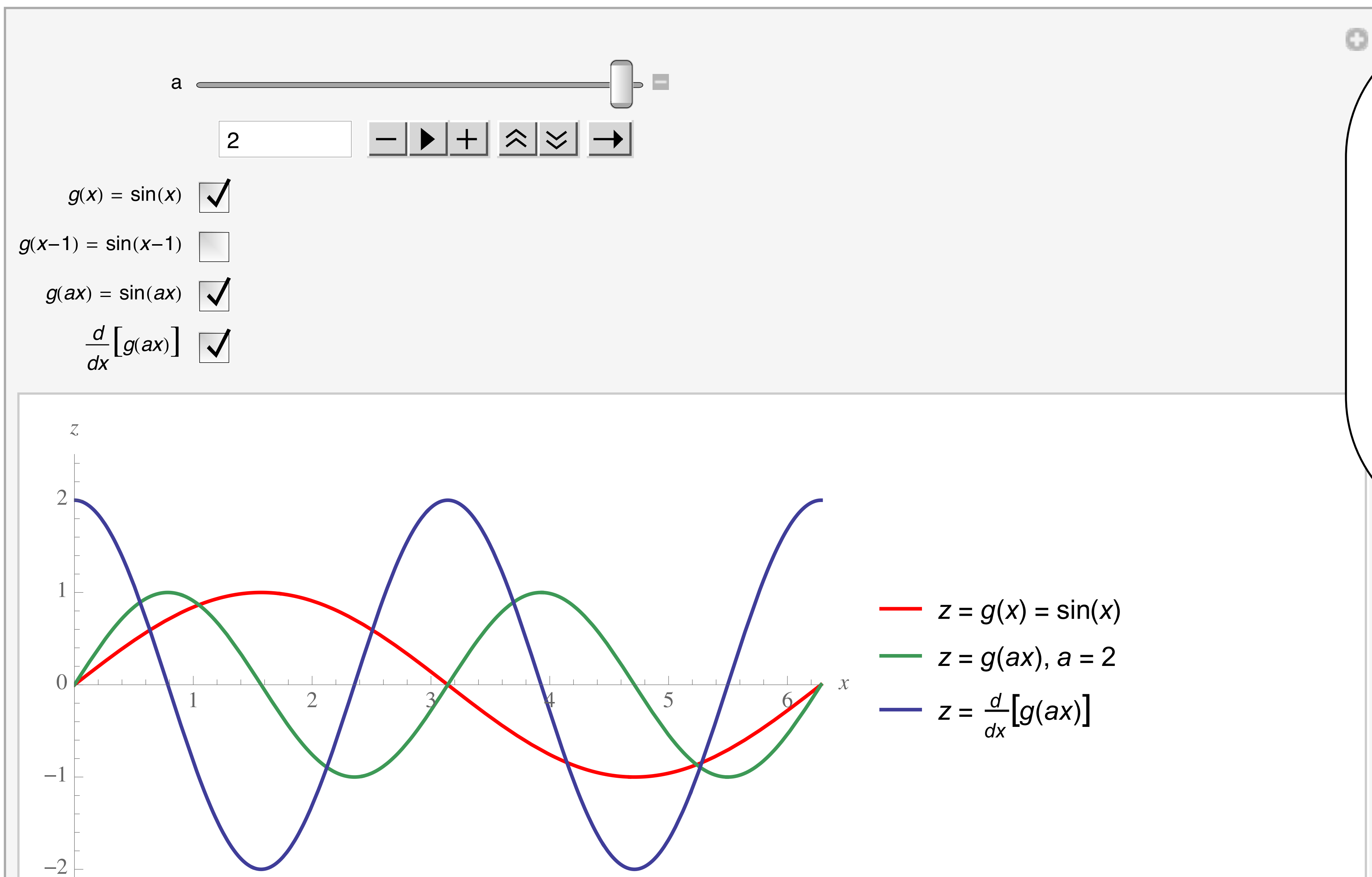
6. Introducing the Chain Rule



"What do you suppose the derivative of $\sin(2x)$ is?"

Applets

6. Introducing the Chain Rule



"Here's a hint: the tangent to $\sin(2x)$ at 0 is *twice as steep* as the tangent to $\sin(x)$ at 0."

Applets

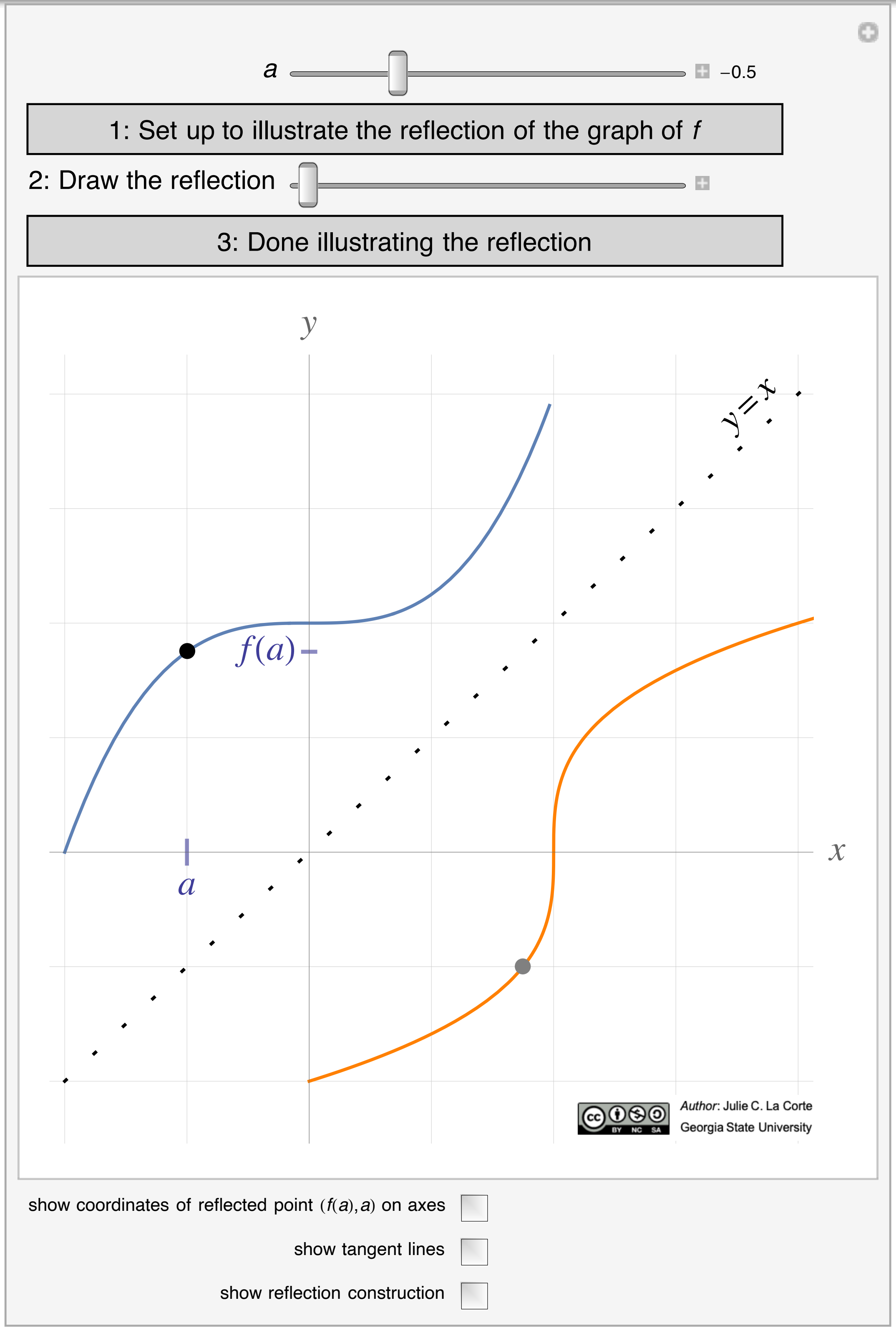
7. Introducing the Inverse Function Theorem with Tangent Lines

This applet provides motivation for the Inverse Function Theorem by building the intuition that if the tangent line to f has slope

$$\frac{\Delta y}{\Delta x}$$

at $(a, f(a))$, then the tangent line to f^{-1} at the corresponding point $(f(a), a)$ ought to have slope

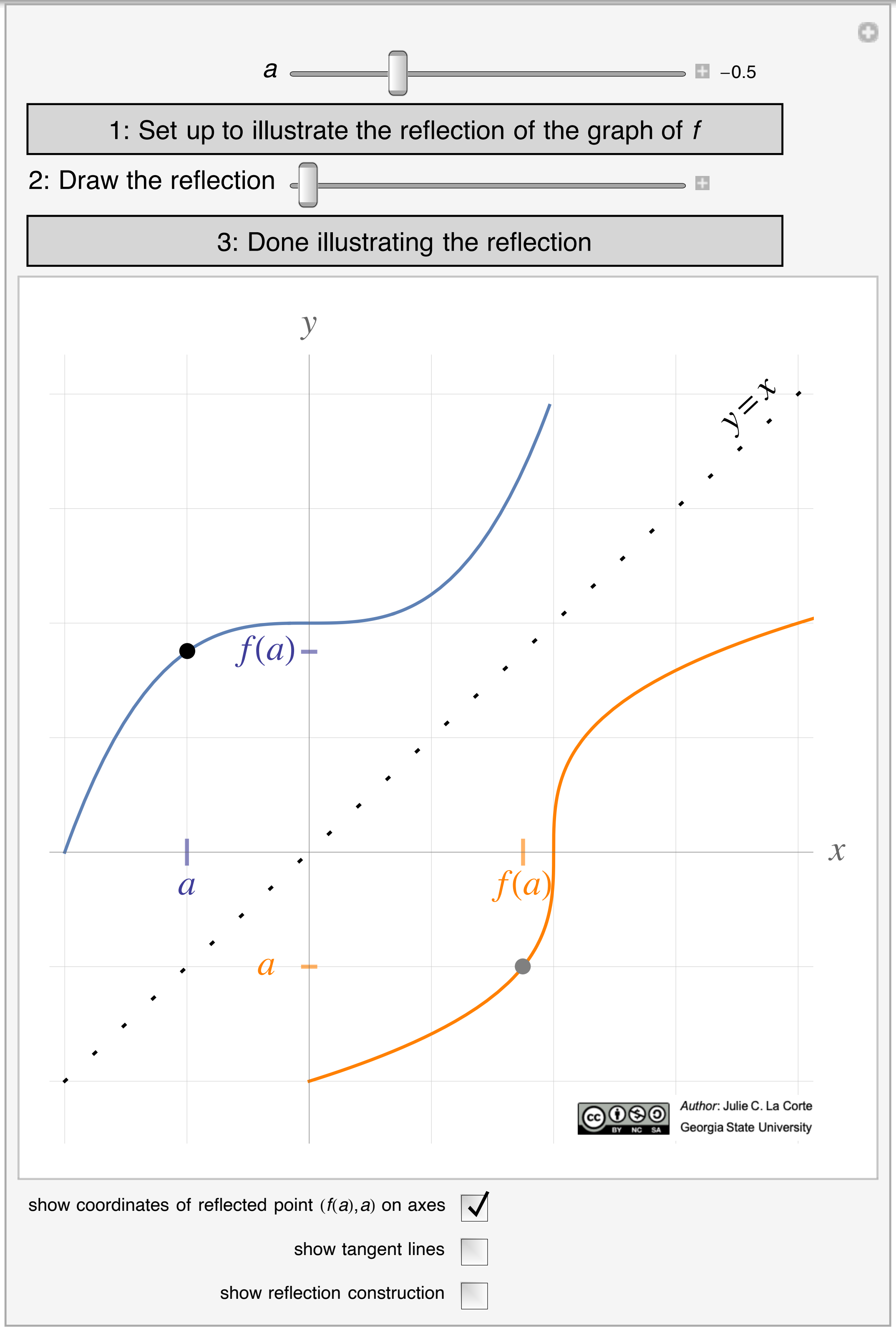
$$\frac{\Delta x}{\Delta y}.$$



Applets

7. Introducing the Inverse Function Theorem with Tangent Lines

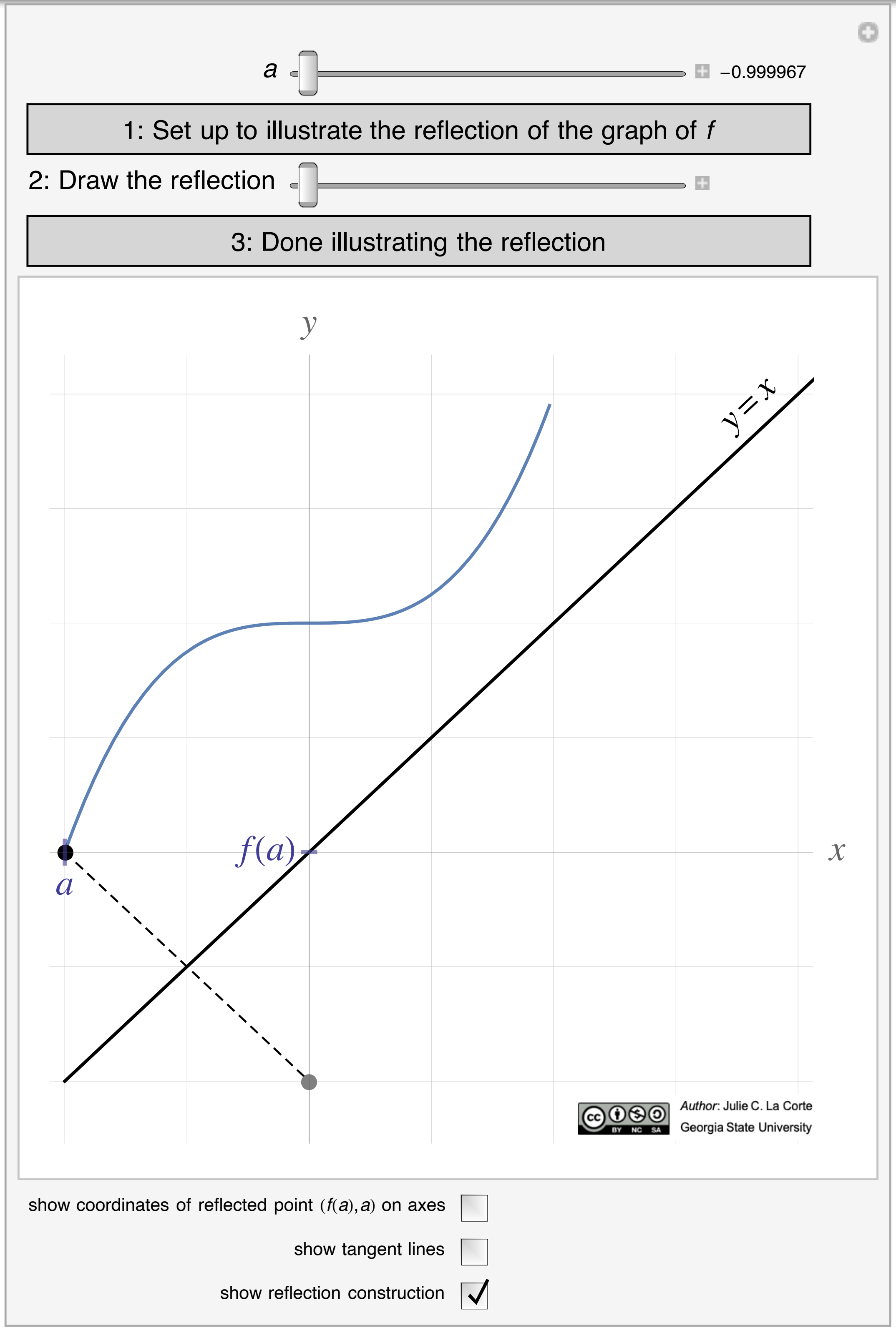
The coordinates of the points $(a, f(a))$ and $(f(a), a)$ can be revealed and hidden using a checkbox.



Applets

7. Introducing the Inverse Function Theorem with Tangent Lines

The applet provides a just-in-time review of the construction of the graph of f^{-1} given the graph of f.



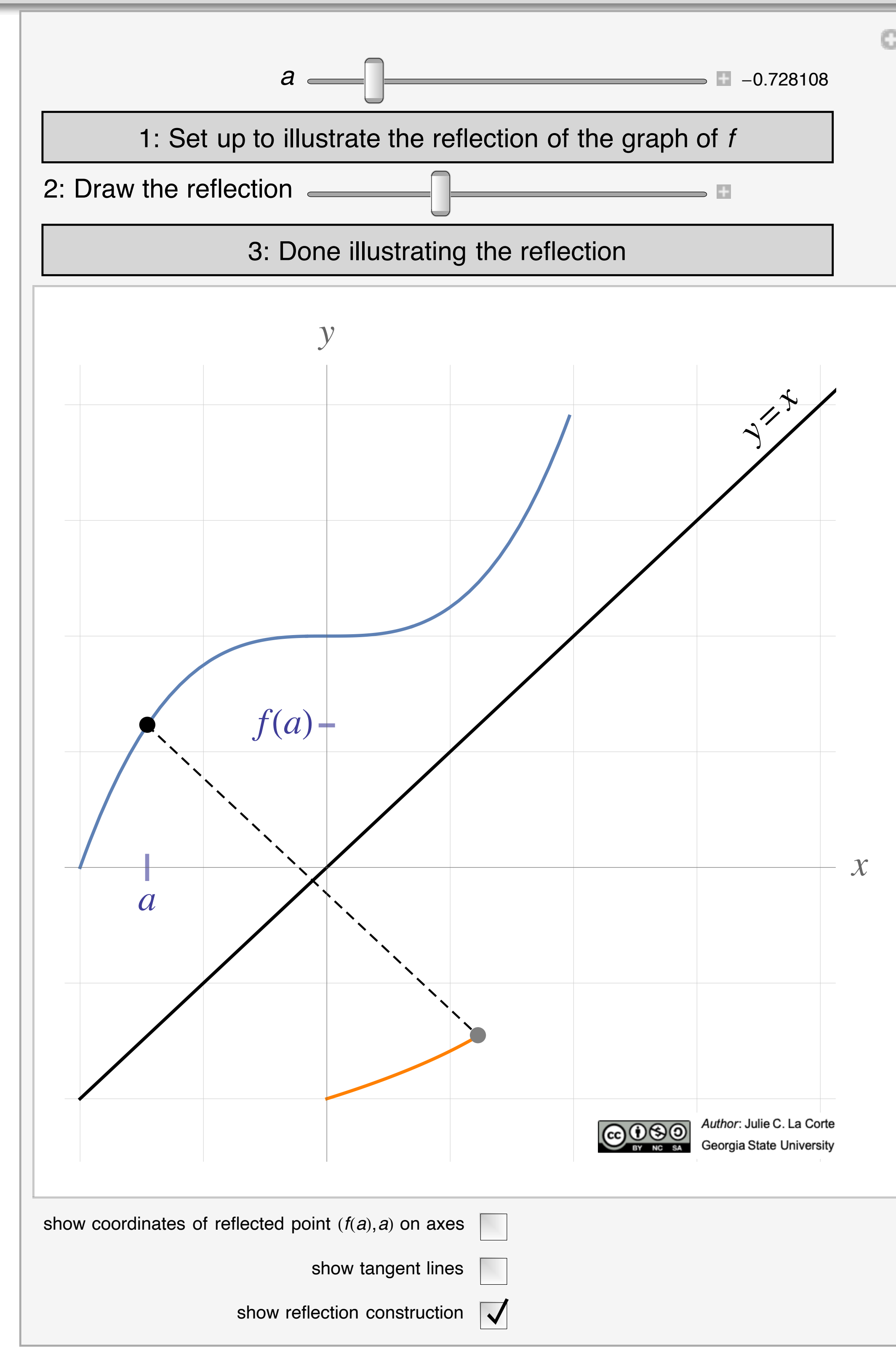
Applets

7. Introducing the Inverse Function Theorem with Tangent Lines

As the slider marked

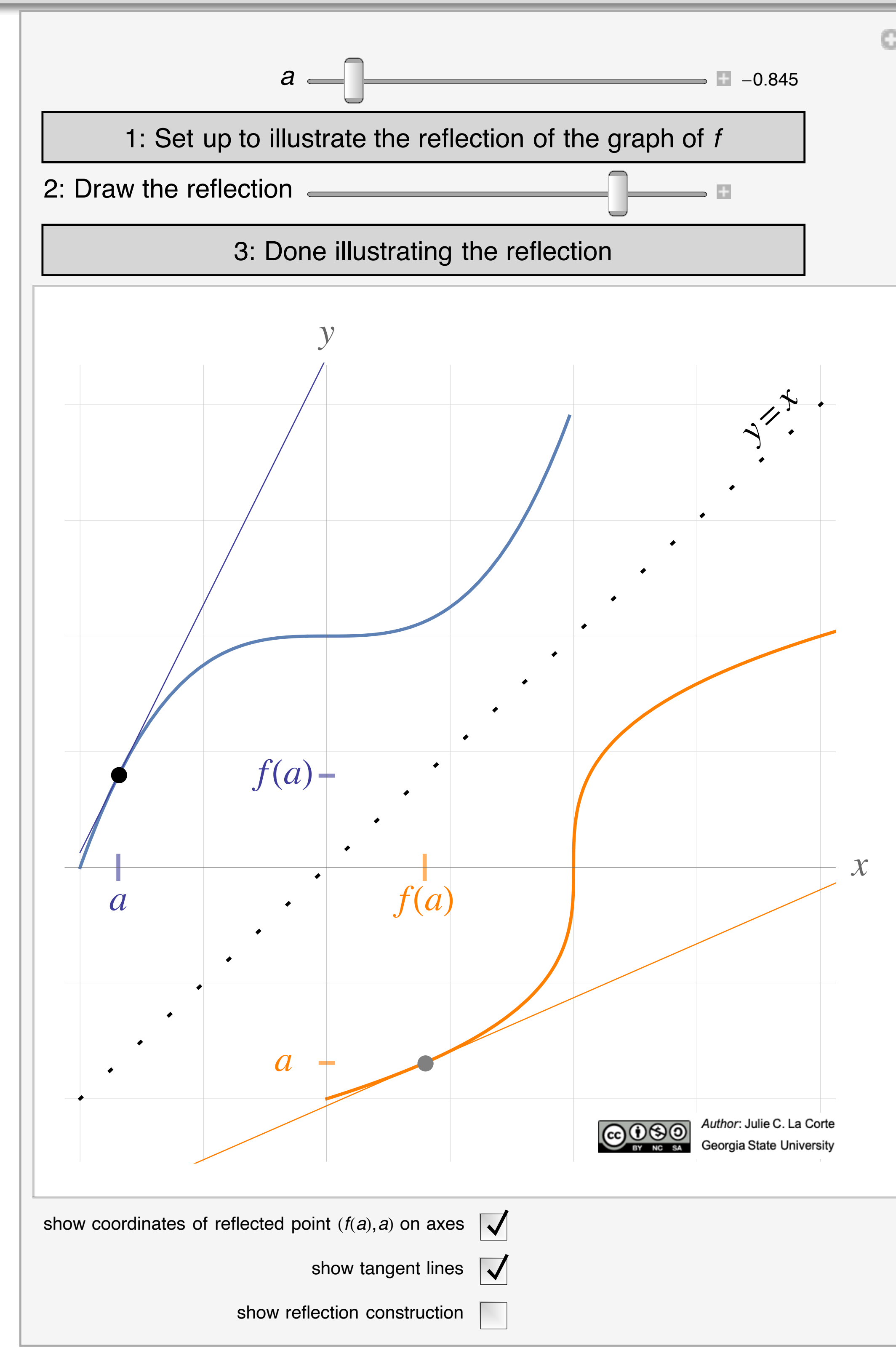
``2: Draw the reflection''

is moved, the graph of f^{-1} appears as if drawn by a pencil at the moving point $(f(a),\ a)$.



7. Introducing the Inverse Function Theorem with Tangent Lines

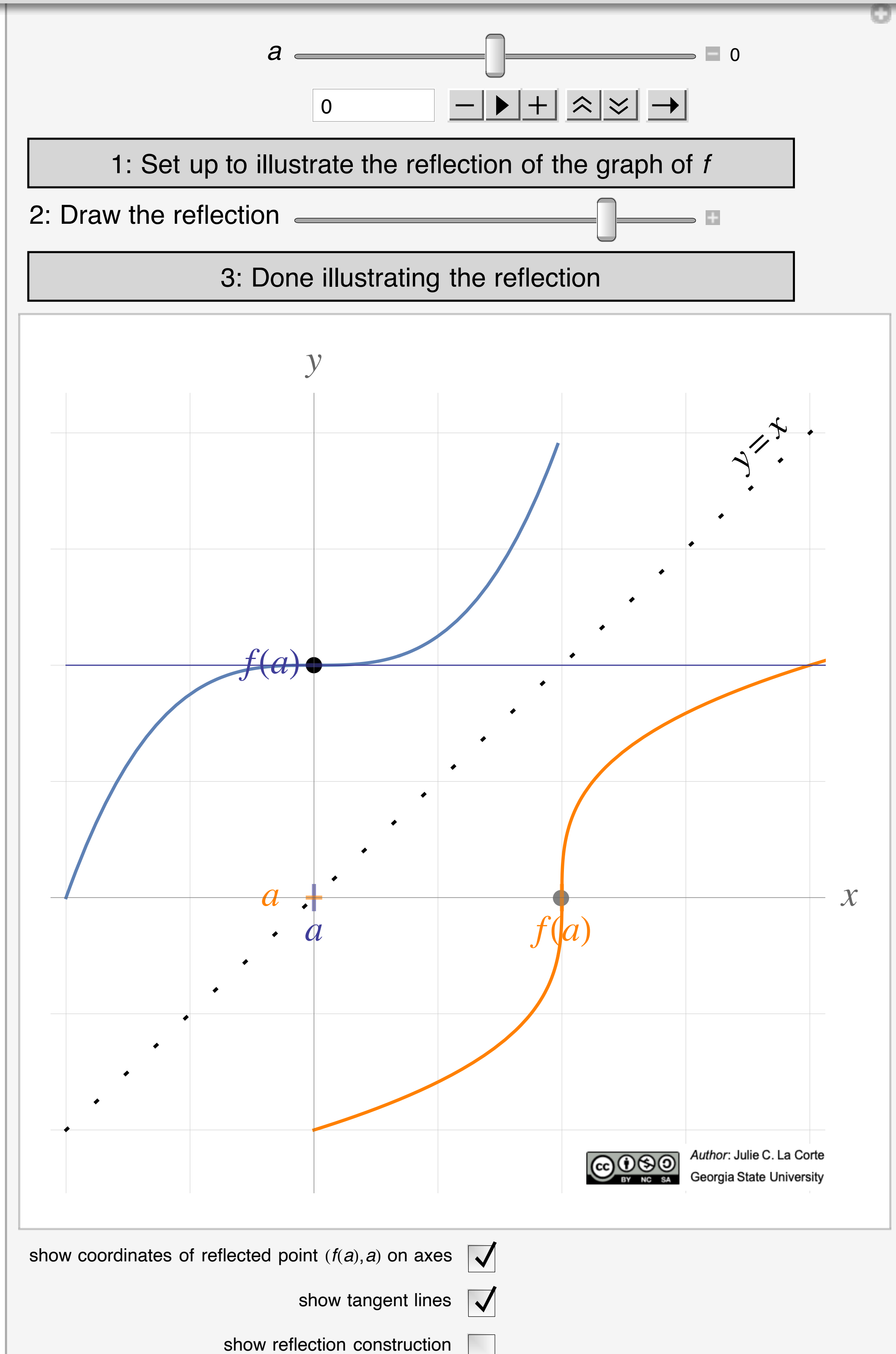
The relationship between the slopes of the tangent lines to the two graphs at coordinate-swapped points is easy to see without introducing any unneeded and potentially confusing notation.



Applets

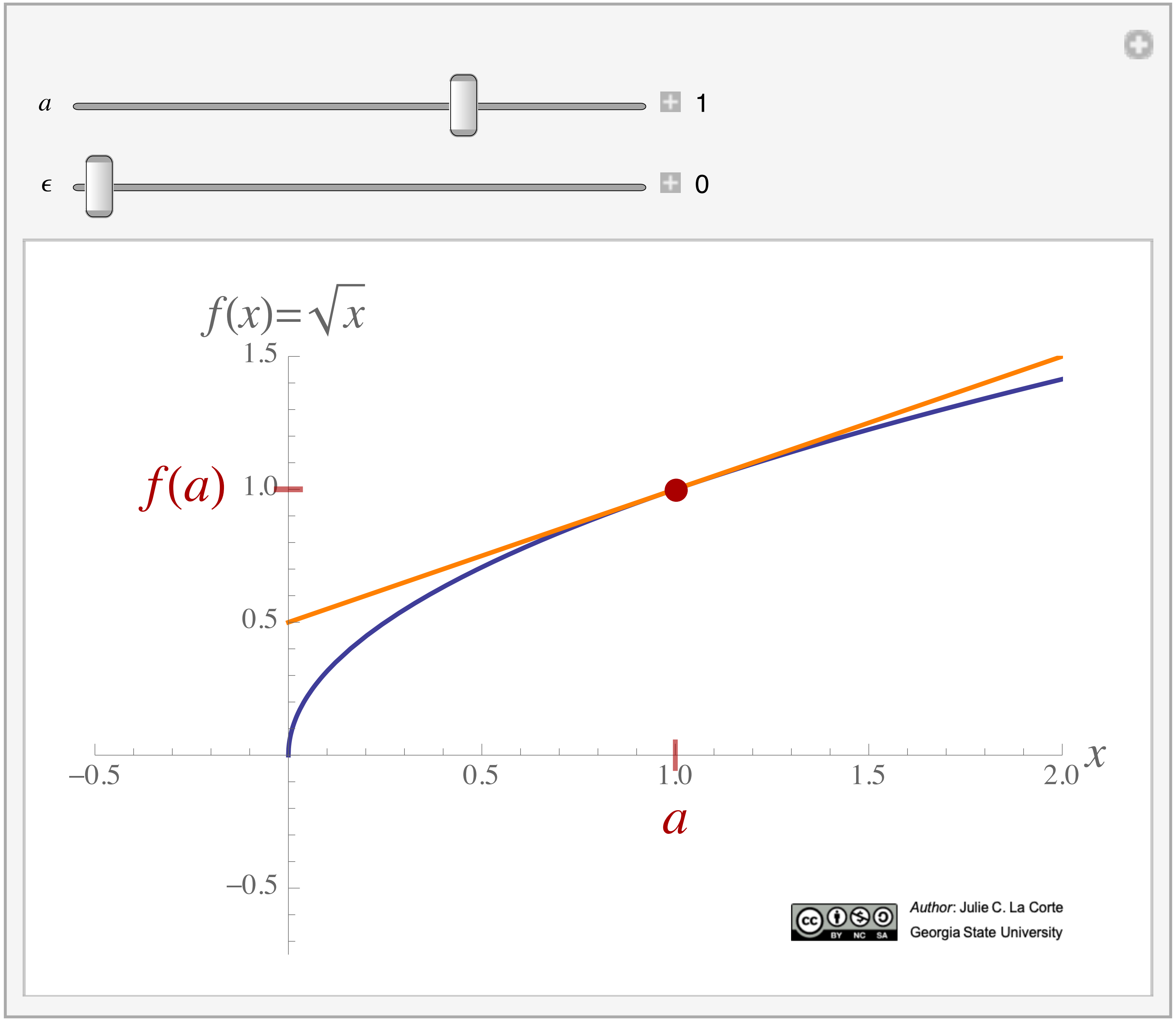
7. Introducing the Inverse Function Theorem with Tangent Lines

We can even see why the Inverse Function Theorem will forbid the tangent to f at a from having a horizontal slope.



Applets

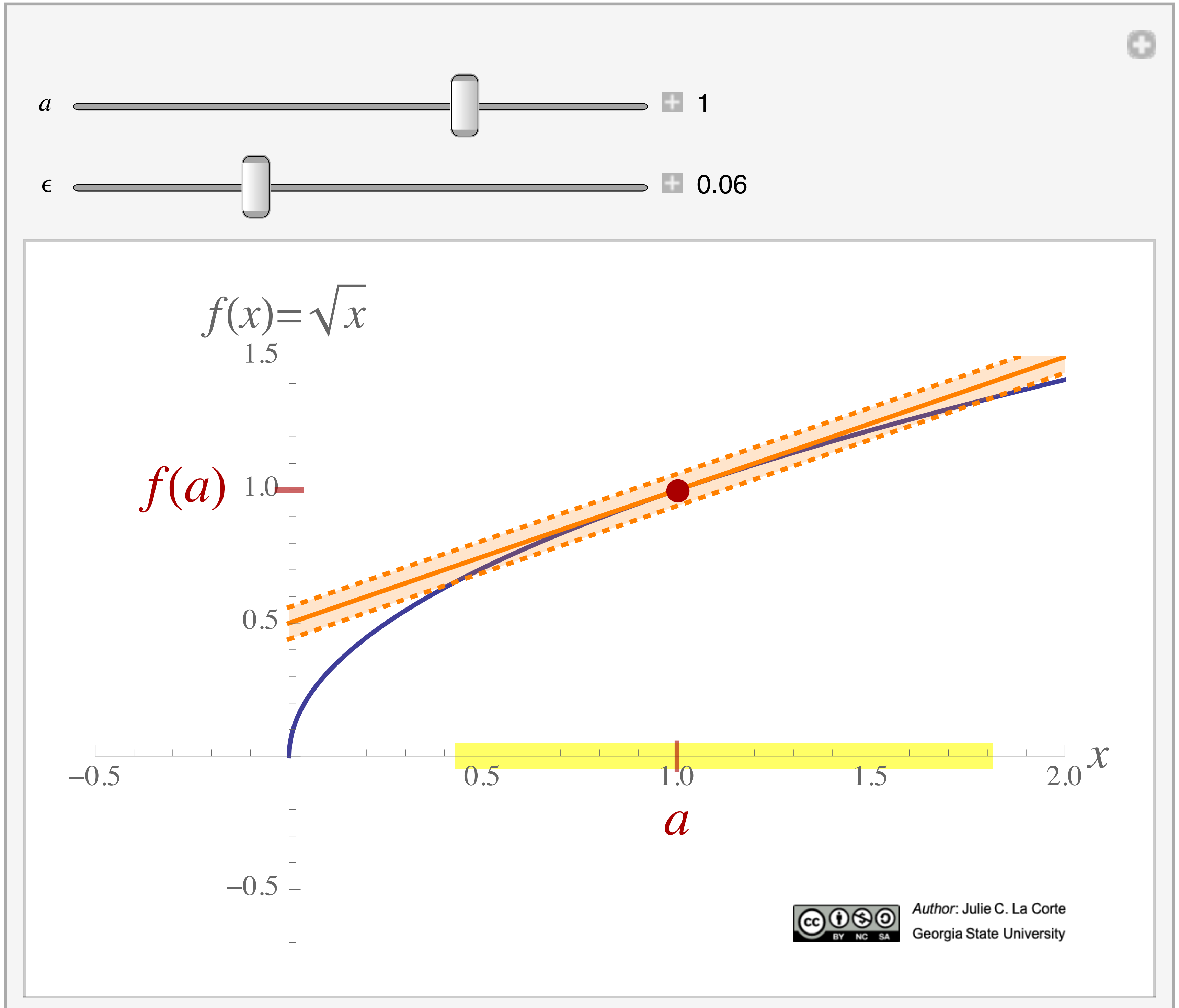
8. Linear Approximation $D_x f$ to a Function of One Variable



Applets allow us to show students the pictures we instructors have in our heads, *without bothering students with unnecessary technical details.*

Applets

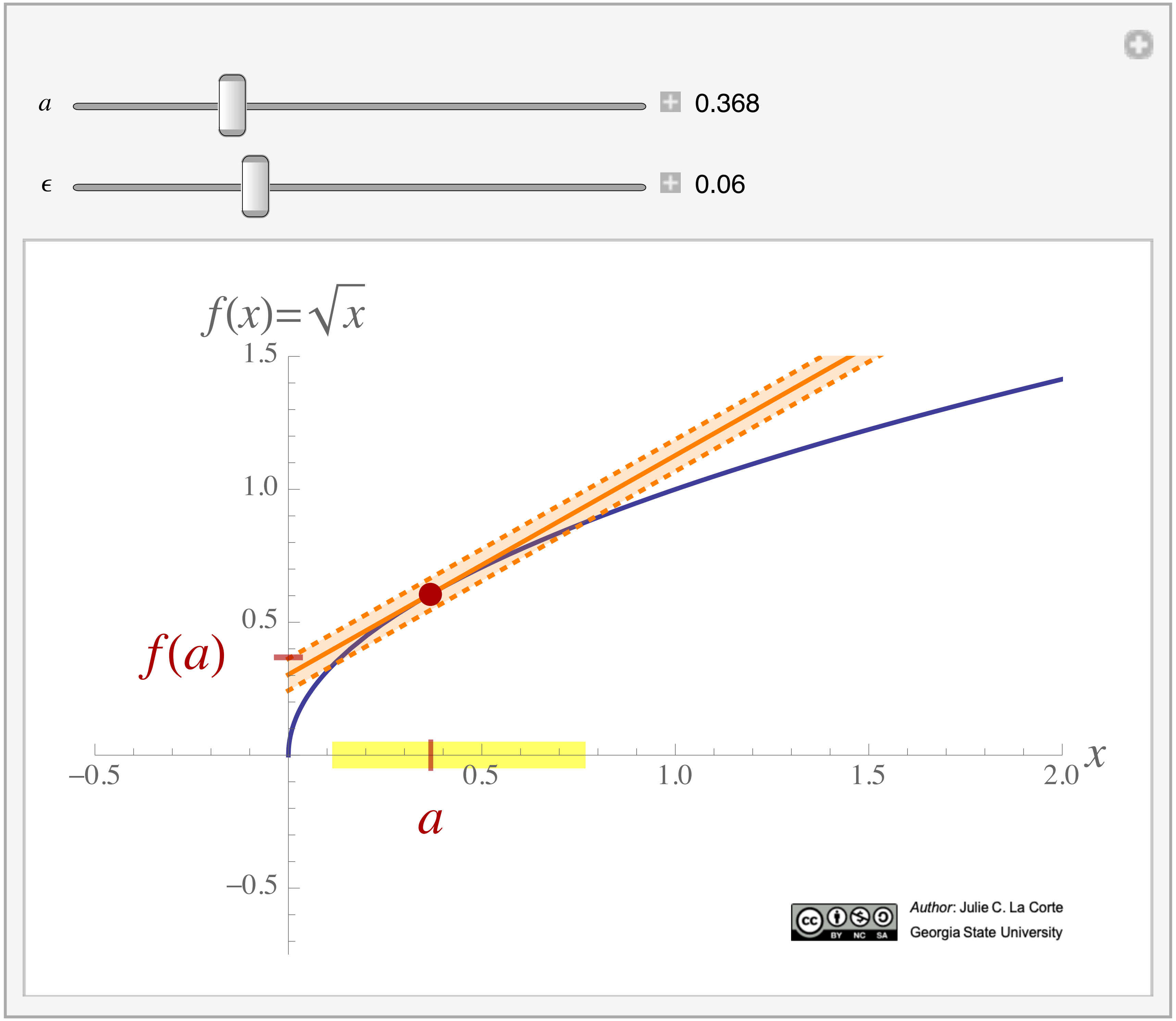
8. Linear Approximation $\mathrm{D}_x f$ to a Function of One Variable



Concepts which are beyond the scope of the course—but nonetheless provide motivation for course material—can be presented informally.

Applets

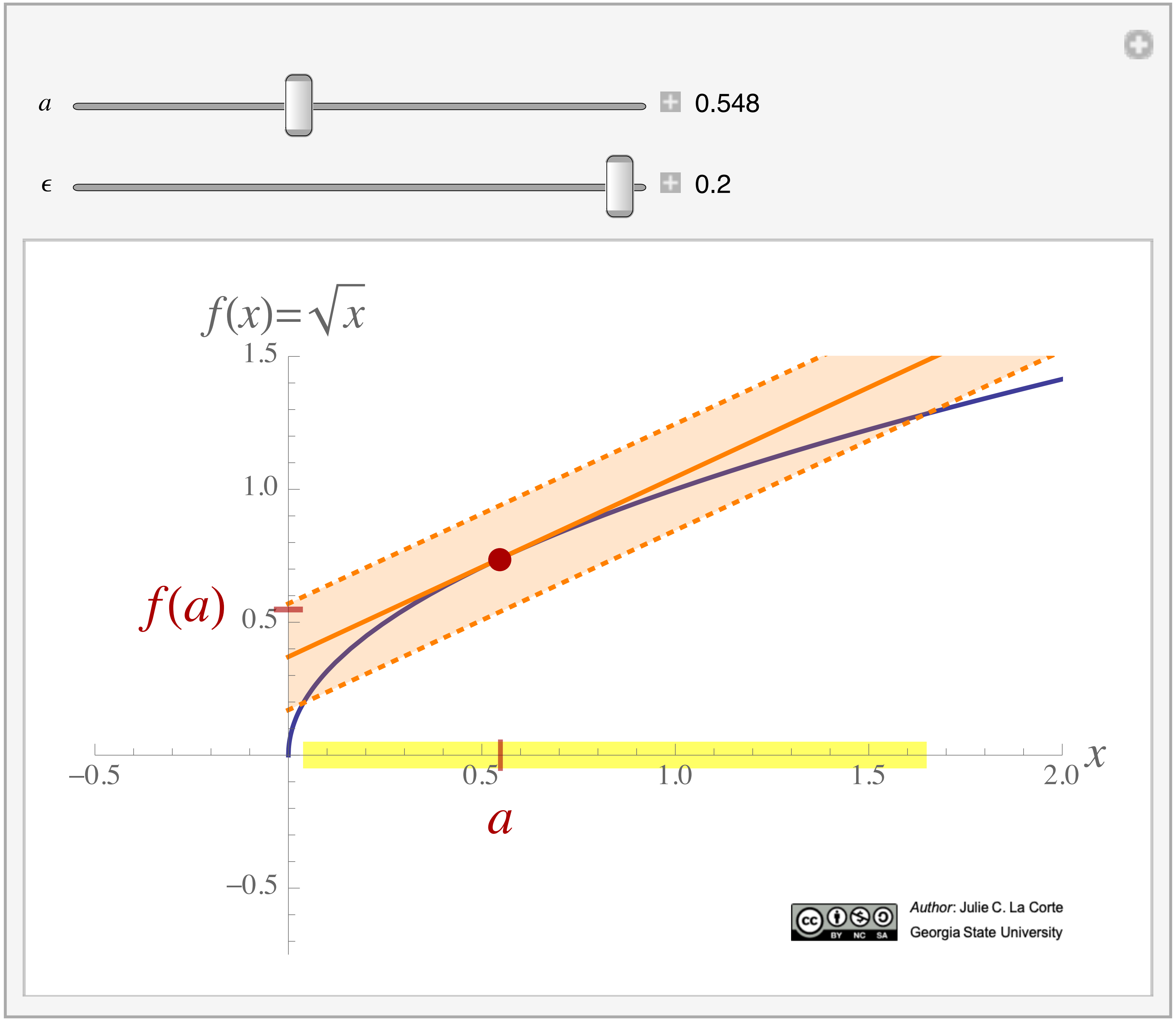
8. Linear Approximation $\mathrm{D}_x f$ to a Function of One Variable



Here the practical difference between linear approximations of the same function *with different tangent points* is illustrated without words.

Applets

8. Linear Approximation $\mathrm{D}_x f$ to a Function of One Variable

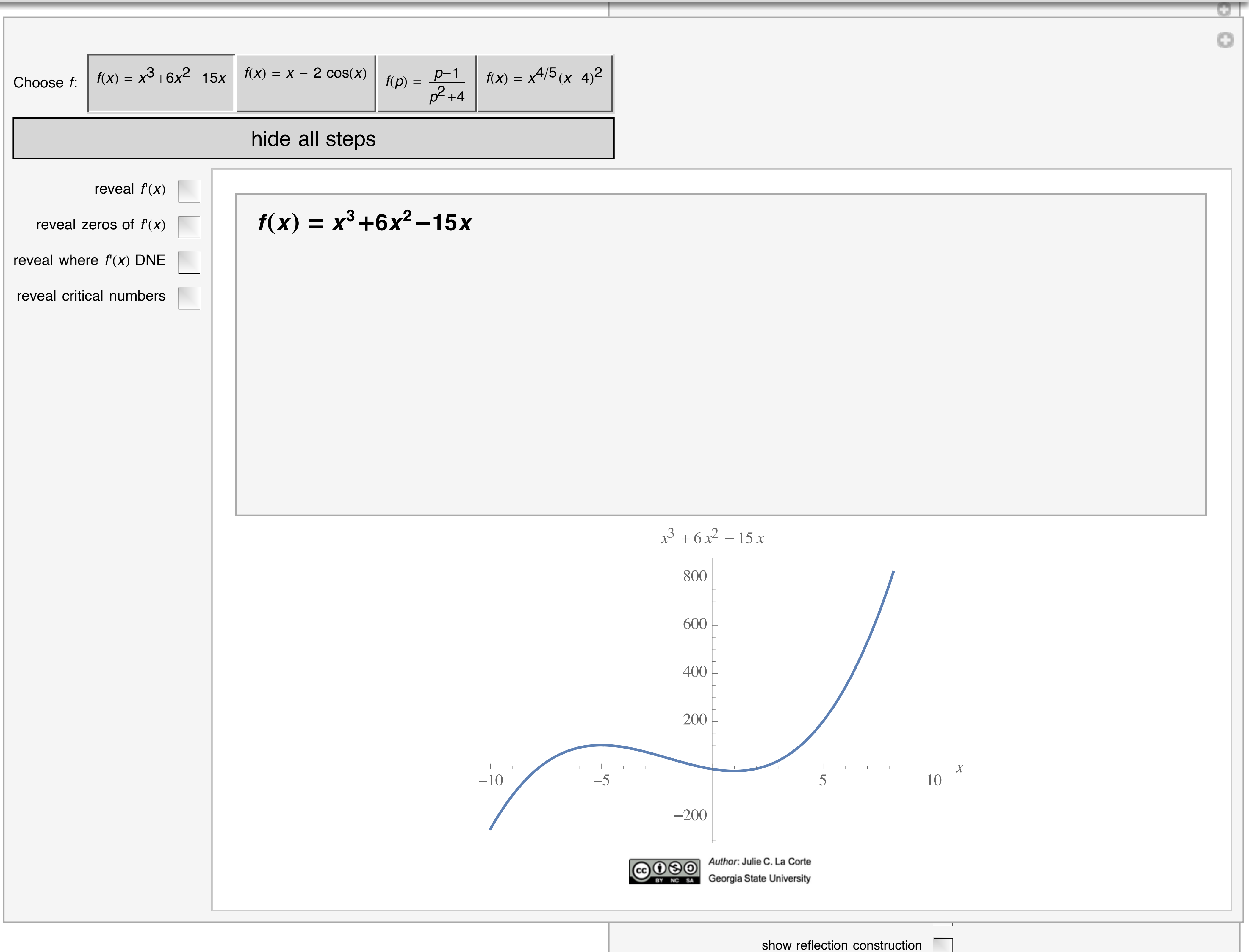


There's no need to define ε-neighborhoods of functions in order to help students *see* that the nearness of (some restriction of) f to its linear approximation depends on the tangent point.

Applets

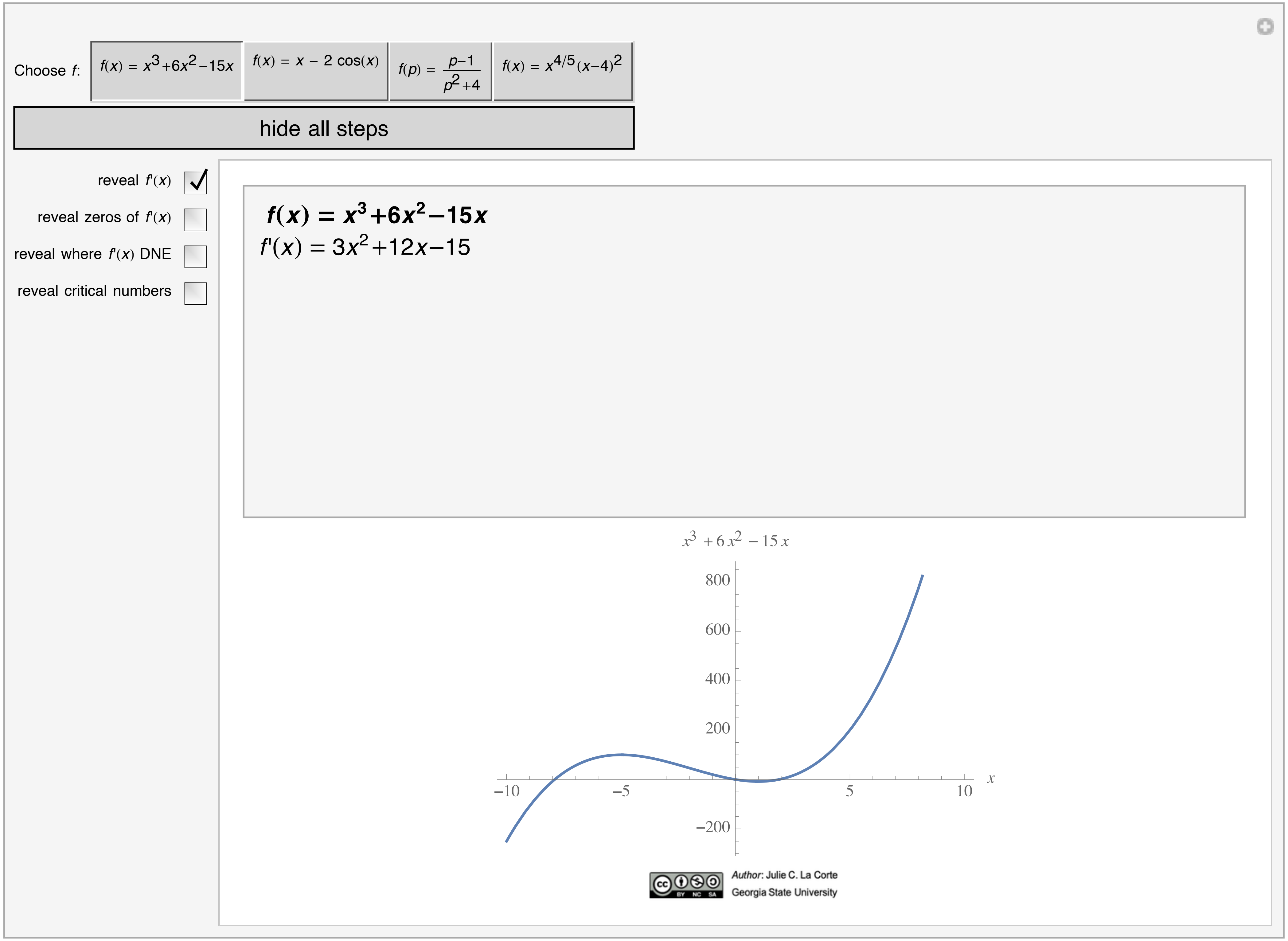
9. Finding Critical Numbers

Here the "Big Picture" strategy for finding critical numbers is presented for four different examples.



9. Finding Critical Numbers

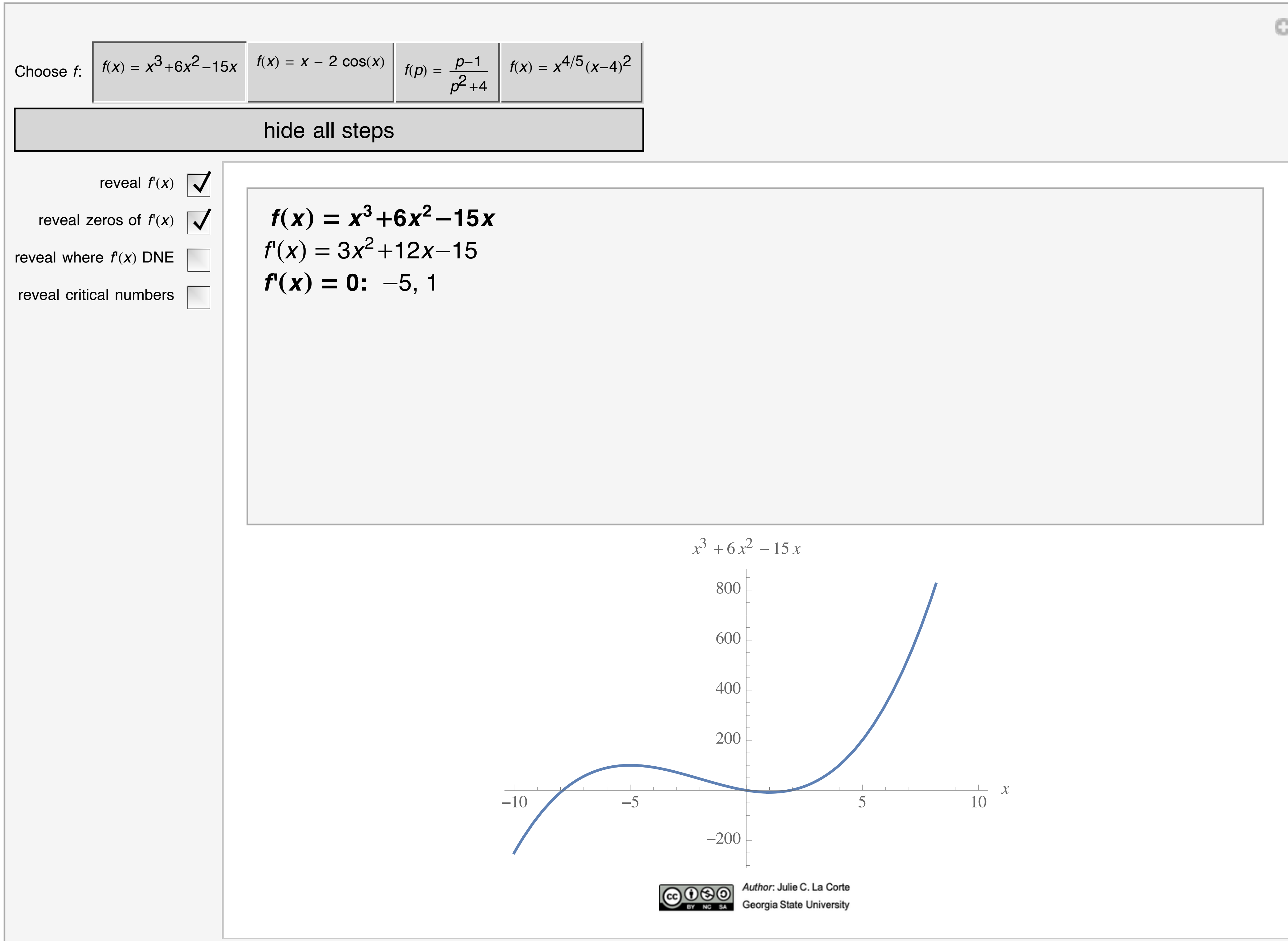
Each step in the process can be revealed and hidden with checkboxes.



Applets

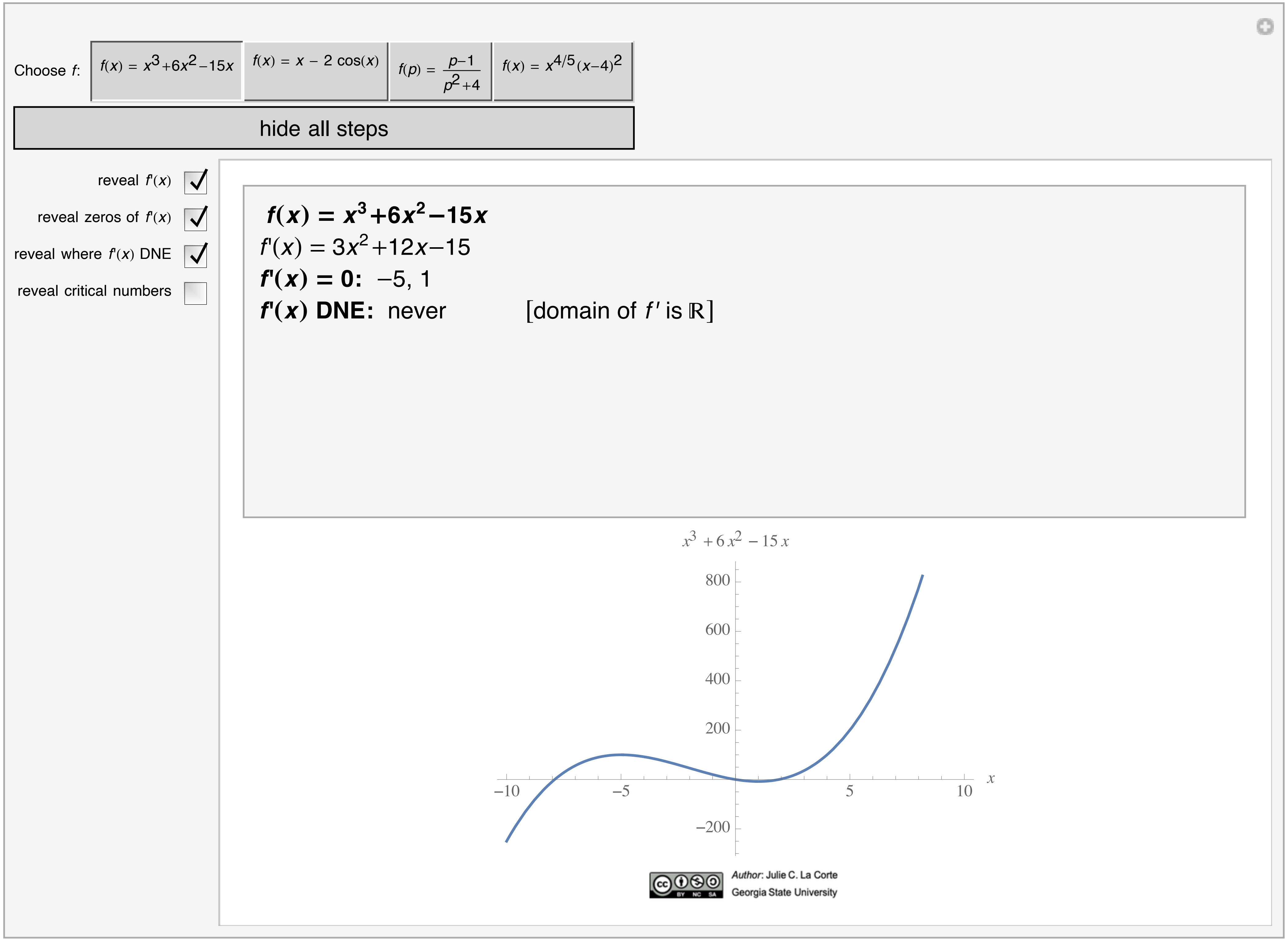
9. Finding Critical Numbers

Each step in the process can be revealed and hidden with checkboxes.



9. Finding Critical Numbers

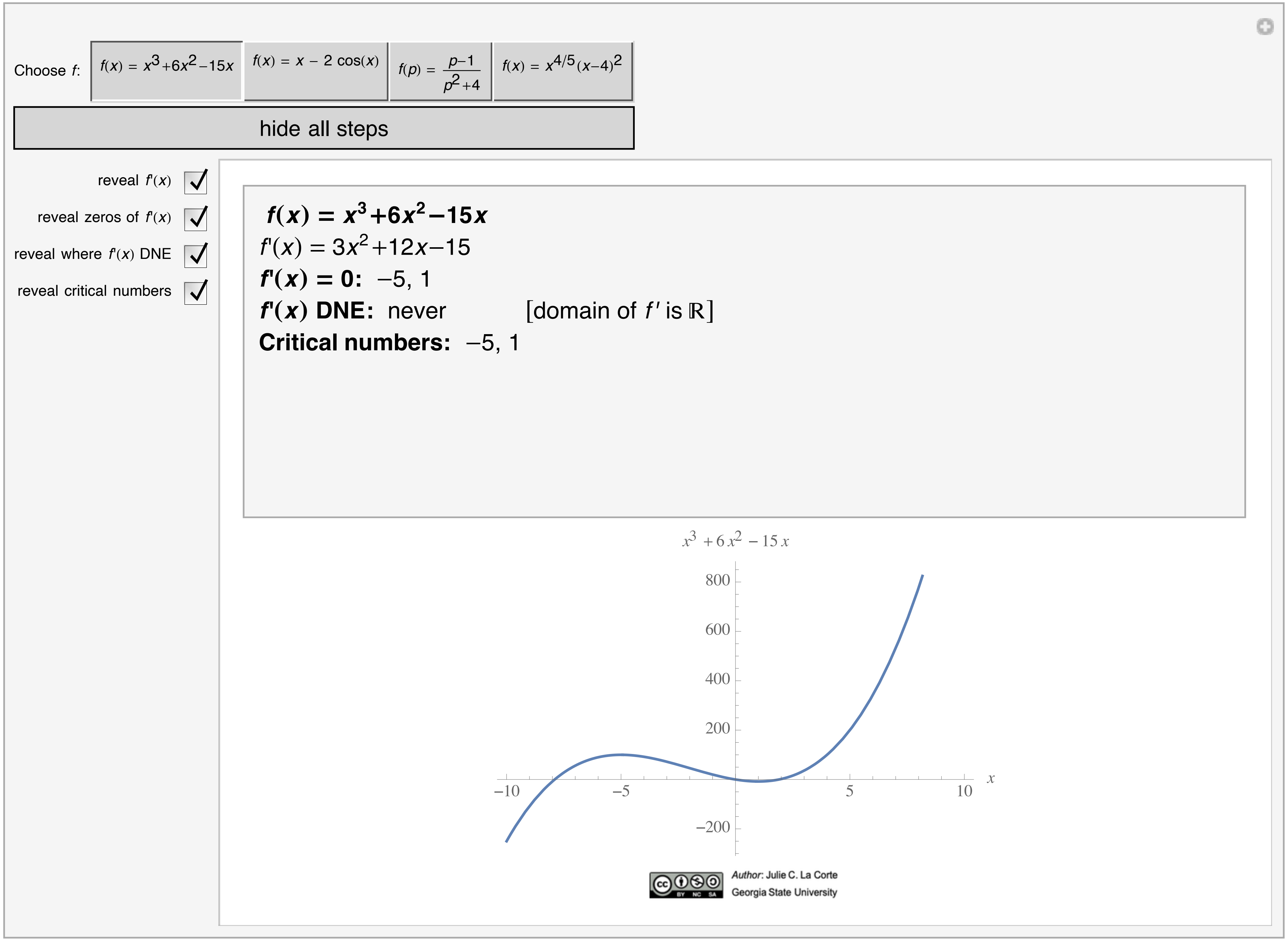
Each step in the process can be revealed and hidden with checkboxes.



Applets

9. Finding Critical Numbers

Each step in the process can be revealed and hidden with checkboxes.

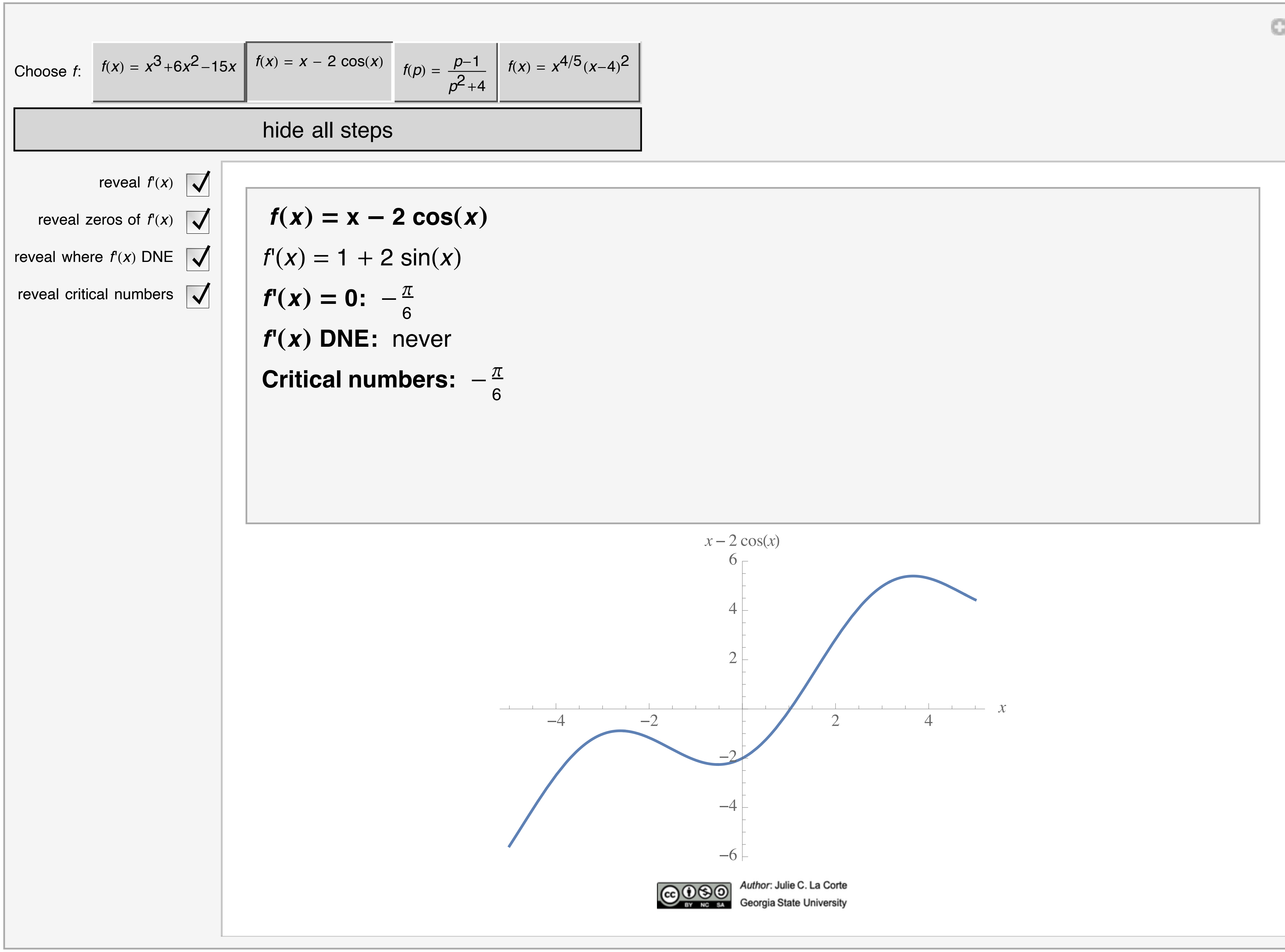


Applets

9. Finding Critical Numbers

The examples chosen each require different technical skills.

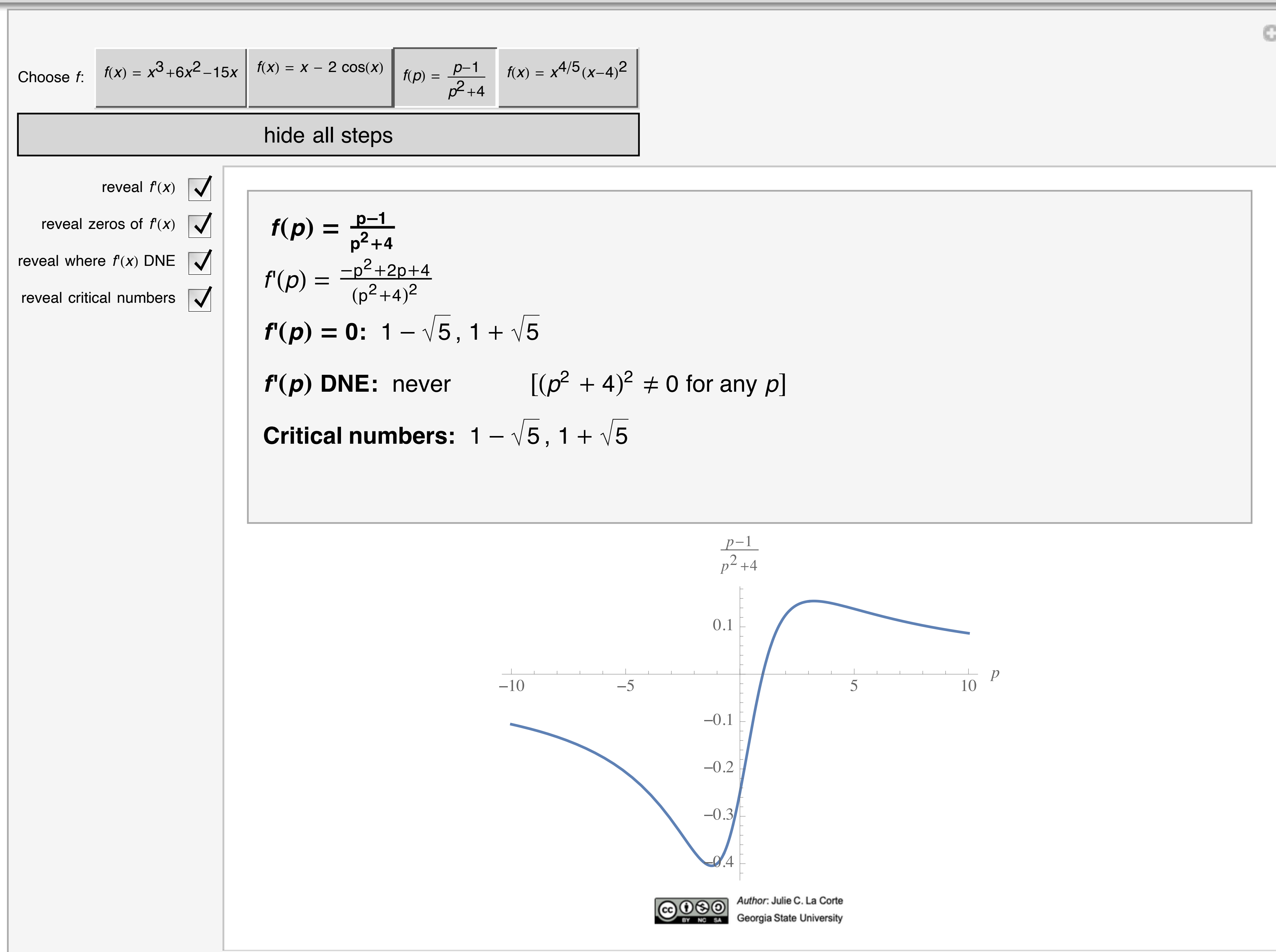
- polynomial
- trigonometric
- rational function
- rational power



Applets

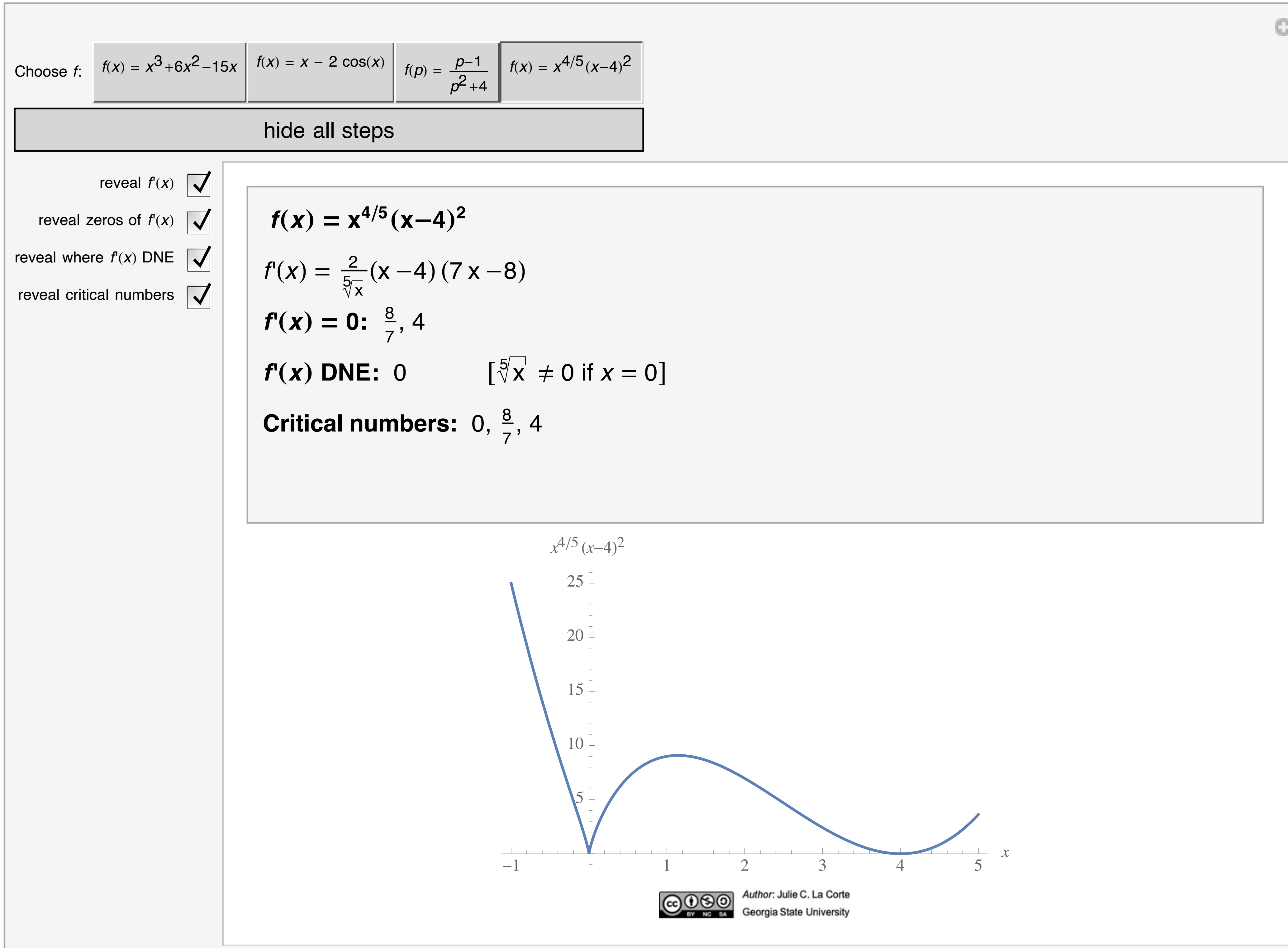
9. Finding Critical Numbers

Students can use these exercises for practice, supplying the missing details...

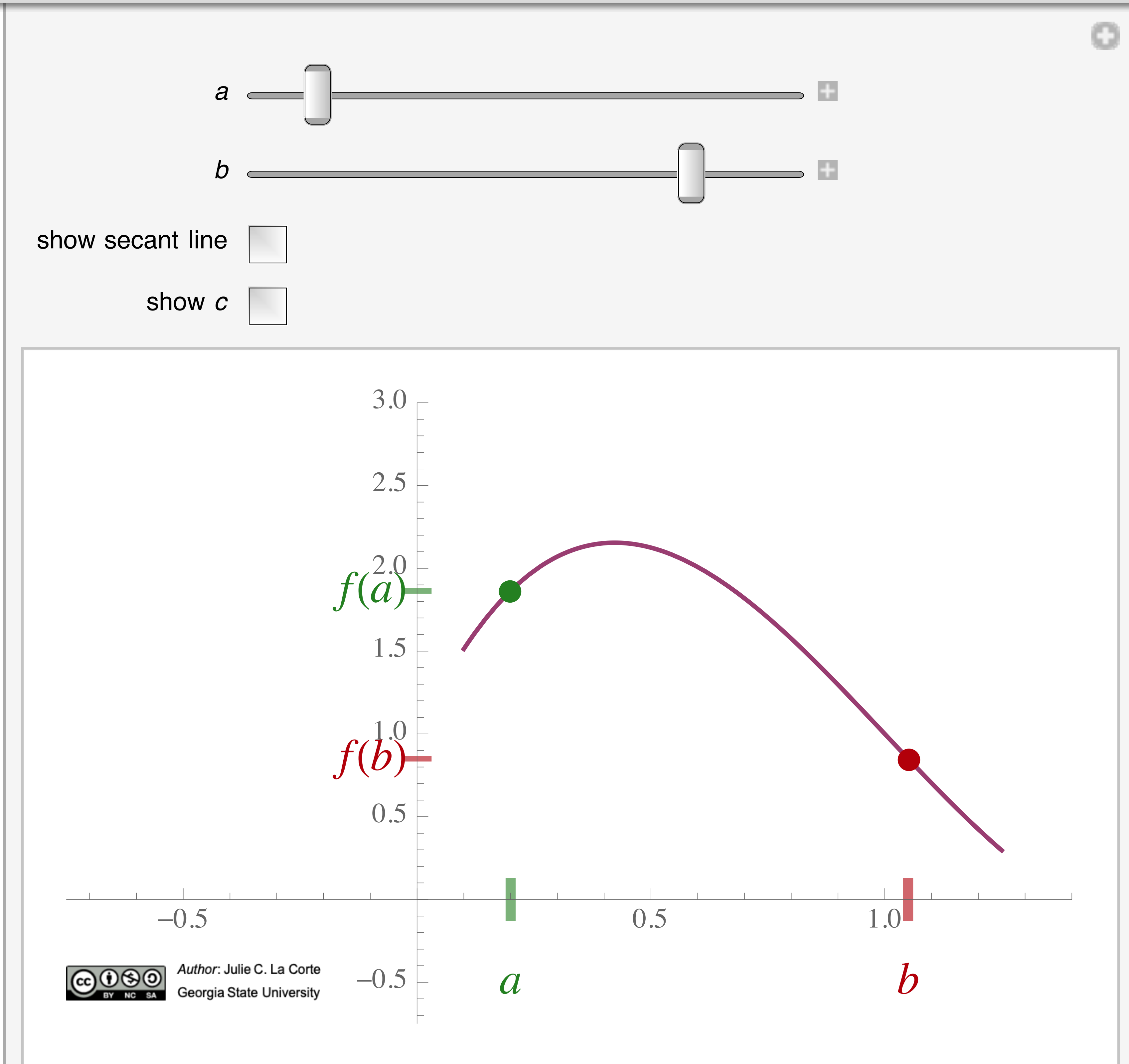


9. Finding Critical Numbers

...all the while staring at an outline of the solution, with the result of each step available on demand.

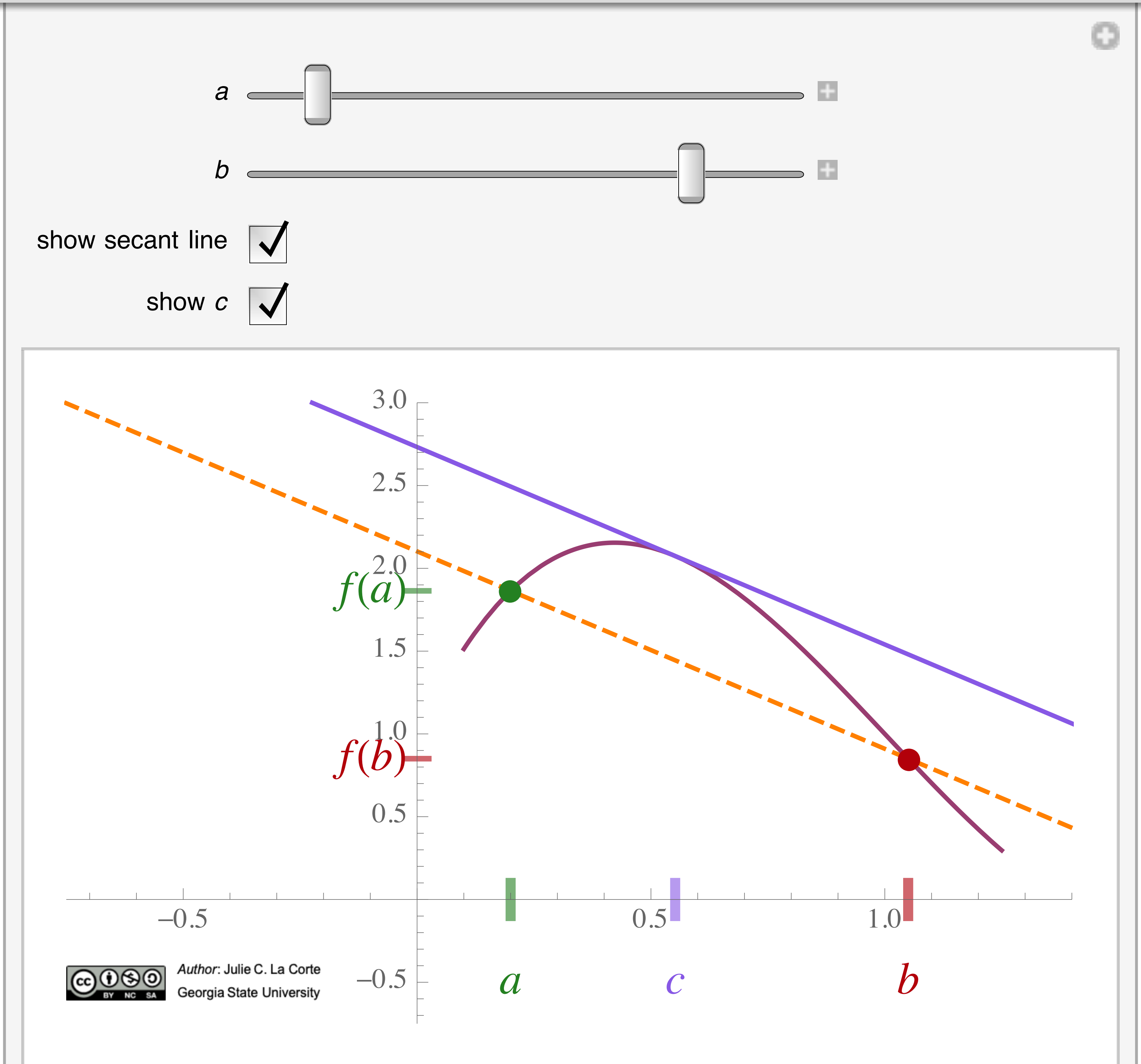


10. Introducing the Mean Value Theorem



It's easier to ***see*** that the slope of the secant line is attained by the derivative at some interior point of $[a,b]$...

10. Introducing the Mean Value Theorem

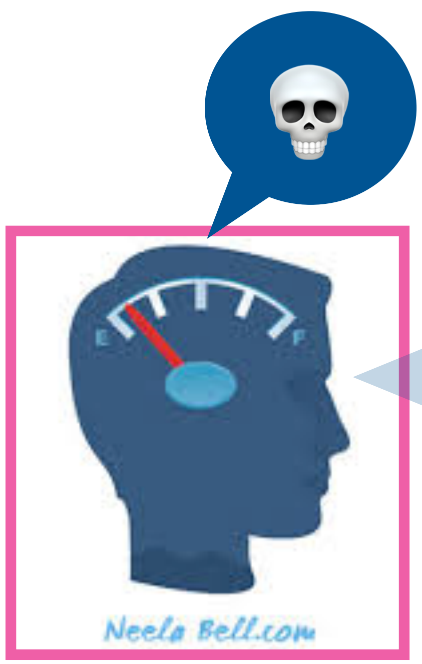


It's easier to ***see*** that the slope of the secant line is attained by the derivative at some interior point of $[a,b]$...

Applets

10. Introducing the Mean Value Theorem

...then it is to ***say*** it using the vocabulary and symbols available to the early Calculus student.

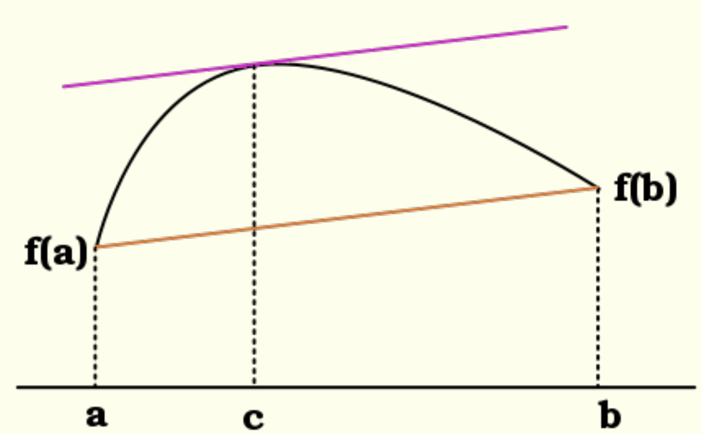


Note that (*) can be rewritten

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

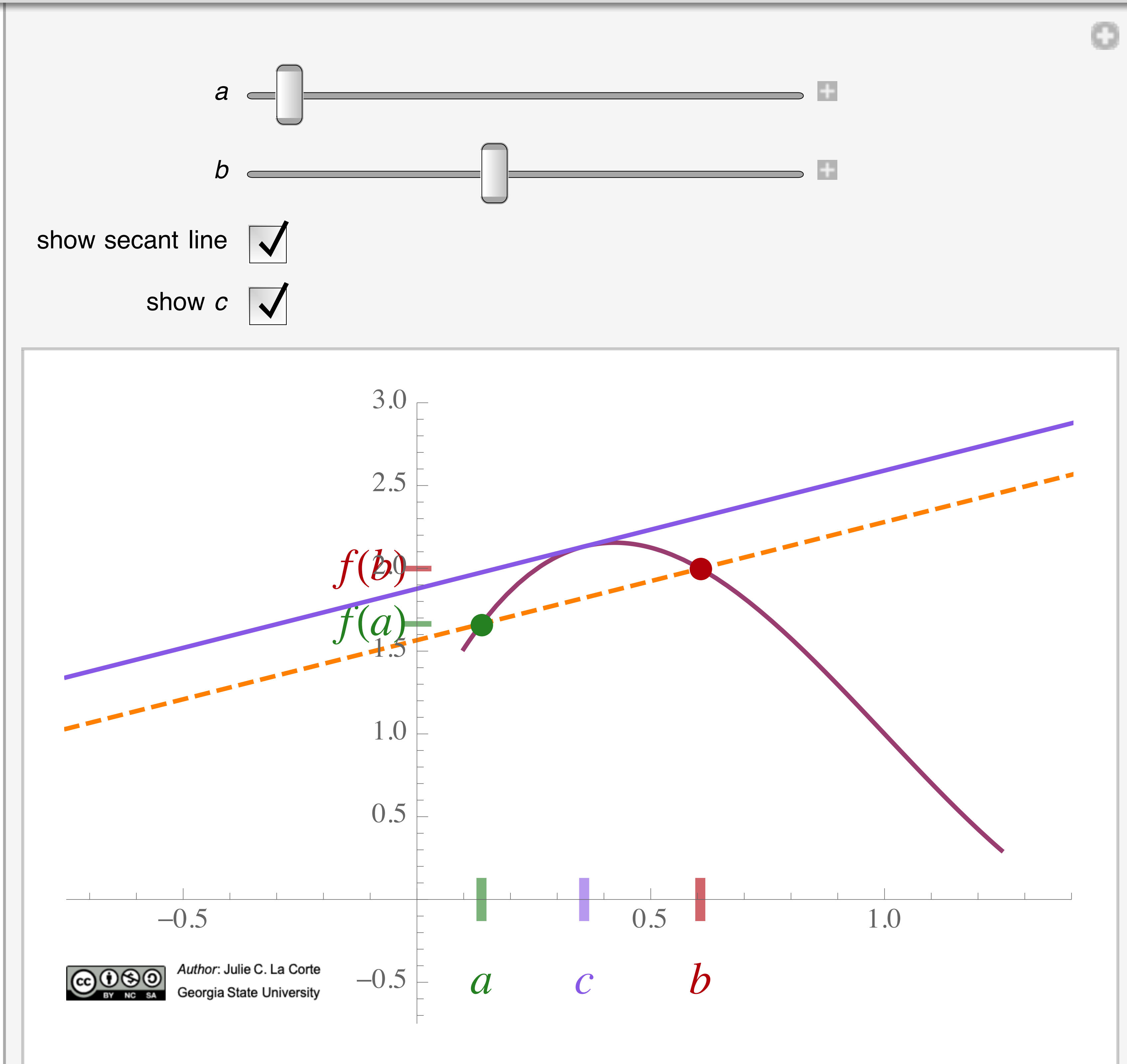
So the Theorem says that:

$$\left(\begin{array}{c}\text{the average}\\ \text{rate of change}\\ \text{over } [a,b]\end{array}\right) = \left(\begin{array}{c}\text{the instantaneous}\\ \text{rate of change}\\ \text{at } c\end{array}\right) \quad \text{for some } c \text{ in the interval } (a,b).$$



Applets

10. Introducing the Mean Value Theorem



By giving the student control over the picture, we empower them to formulate questions they might otherwise struggle to articulate.

Applets

11. Intervals of Increase/Decrease

Finding intervals of increase/decrease was another known problem spot for our students.

Ex. 1. Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing, and the intervals on which it is decreasing.

Solution.

Step 1. Find the critical numbers and mark them on the number line.

Critical numbers happen where $f'(x) = 0$ or $f'(x)$ is undefined.

$f'(x) = 0$:

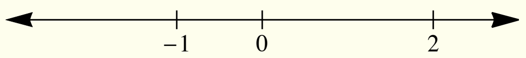
$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

We see that $f'(x) = 0$ when $x = 0$, $x = 2$, or $x = -1$.

$f'(x)$ is undefined:

This never happens, because f is a polynomial (and therefore its domain is all real numbers).

Critical numbers on the number line:



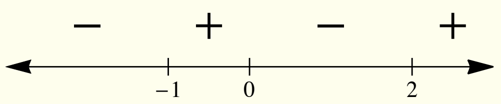
Step 2. Determine the sign of $f'(x)$ in each interval.

We've already factored $f'(x) = 12x(x-2)(x+1)$.

This makes it easy to see where $f'(x)$ is positive or negative—we can just find where each *factor* is positive and negative, and then count the negative signs.

- An odd number of negative factors (e.g. positive × negative) yields a negative number.
- An even number of negative factors (e.g. positive × positive) yields a positive number.

interval	sign of $f'(x)$	$12x$	$(x-2)$	$(x+1)$
$-\infty < x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$+$	$-$	$-$	$+$
$0 < x < 2$	$-$	$+$	$-$	$+$
$2 < x < \infty$	$+$	$+$	$+$	$+$



Step 3. Apply Increasing/Decreasing Test.

f is decreasing on $(-\infty, -1)$ and $(0, 2)$, increasing on $(-1, 0)$ and $(2, \infty)$.

(We could use closed *or* open intervals here, because f' exists at each critical number.)

Applets

11. Intervals of Increase/Decrease

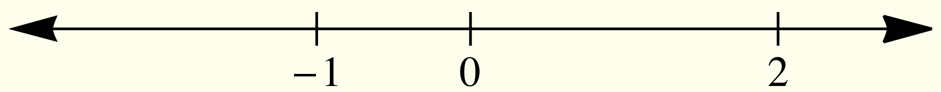
Finding intervals of increase/decrease was another known problem spot for our students.

Ex. 1. Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing, and the intervals on which it is decreasing.

11. Intervals of Increase/Decrease

The process of identifying intervals where $\operatorname{sign}(f')$=const can be made tactile and visual using an applet.

Critical numbers on the number line:



STEP 2. Determine the sign of $f'(x)$ in each interval.

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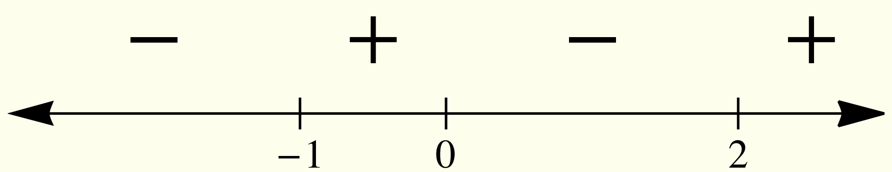
- An odd number of negative factors (e.g. POSITIVE × NEGATIVE) yields a neg
- An even number of negative factors (e.g. POSITIVE × POSITIVE) yields a posi

interval	sign of $f'(x)$	$12x$	$(x-2)$	$(x +$
$-\infty < x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$+$	$-$	$-$	$+$
$0 < x < 2$	$-$	$+$	$-$	$+$
$2 < x < \infty$	$+$	$+$	$+$	$+$



STEP 3. Apply Increasing/Decreasing Test.

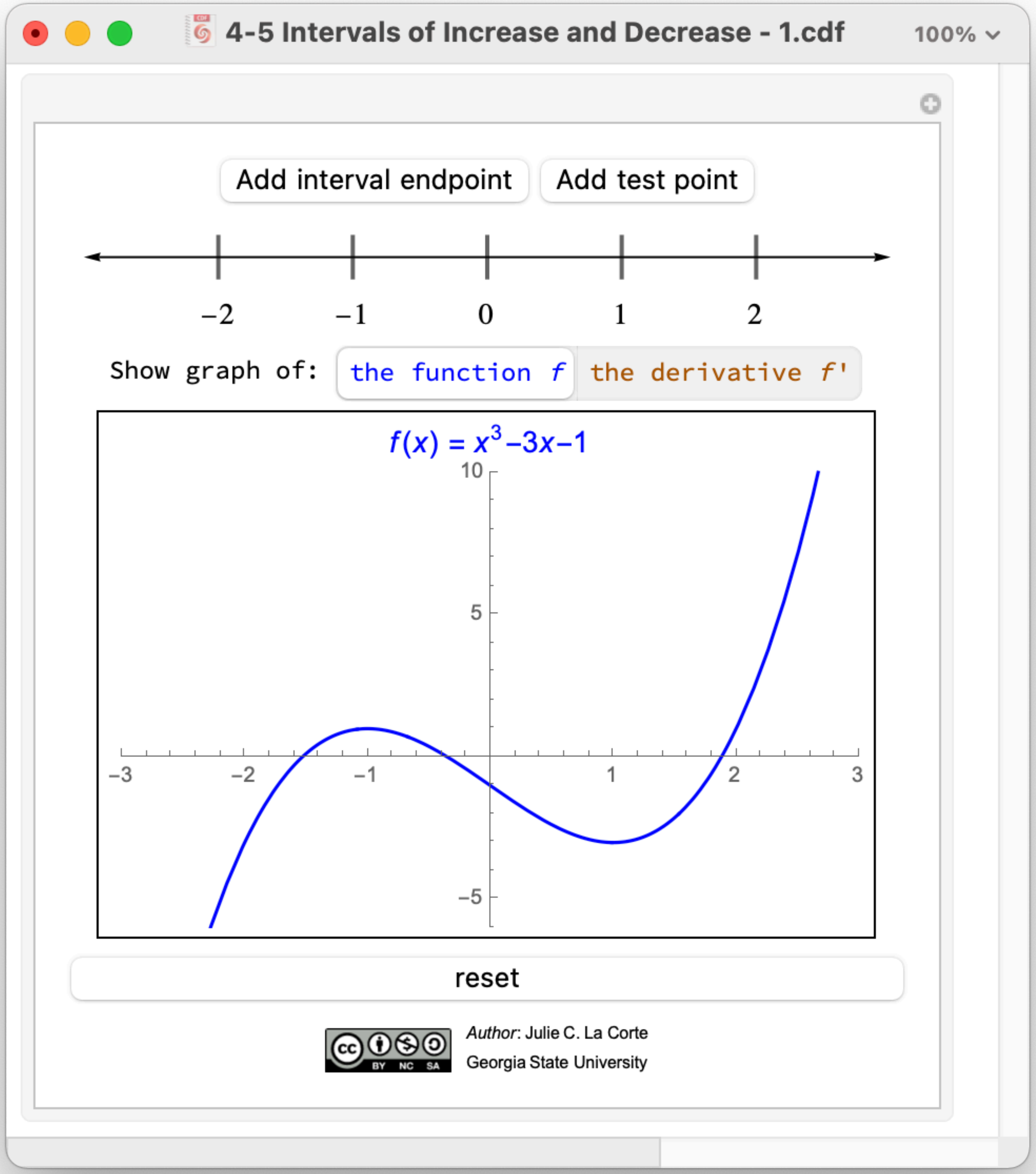
f is decreasing on $(-\infty,-1)$ and $(0,2)$, increasing on $(-1,0)$ and $(2,$

(We could use closed *or* open intervals here, because f' exists at each c

Applets

11. Intervals of Increase/Decrease

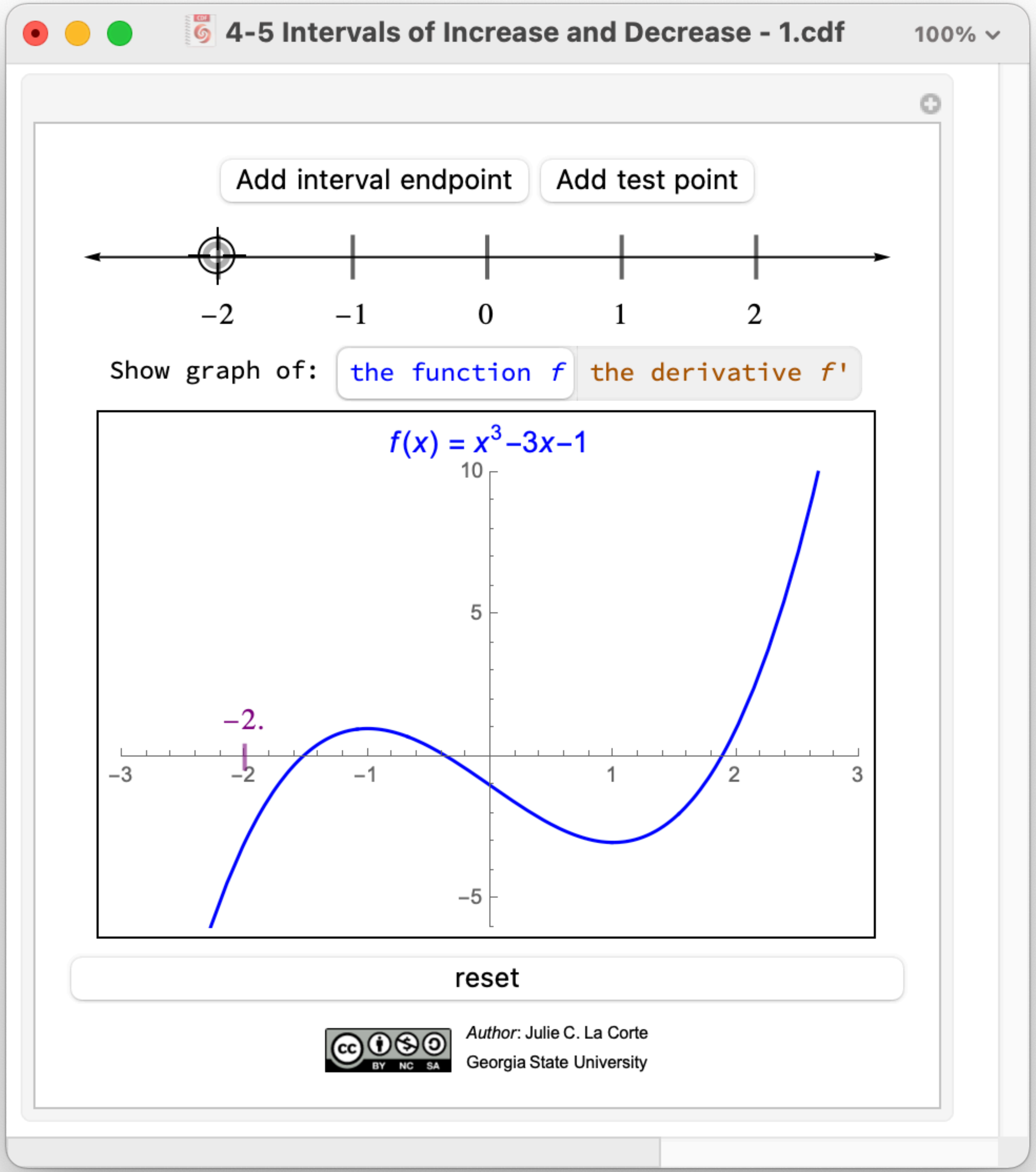
In this applet, students are presented with a number line representing x-values (*top*) and buttons that allow them to add interval endpoints and test points.



Applets

11. Intervals of Increase/Decrease

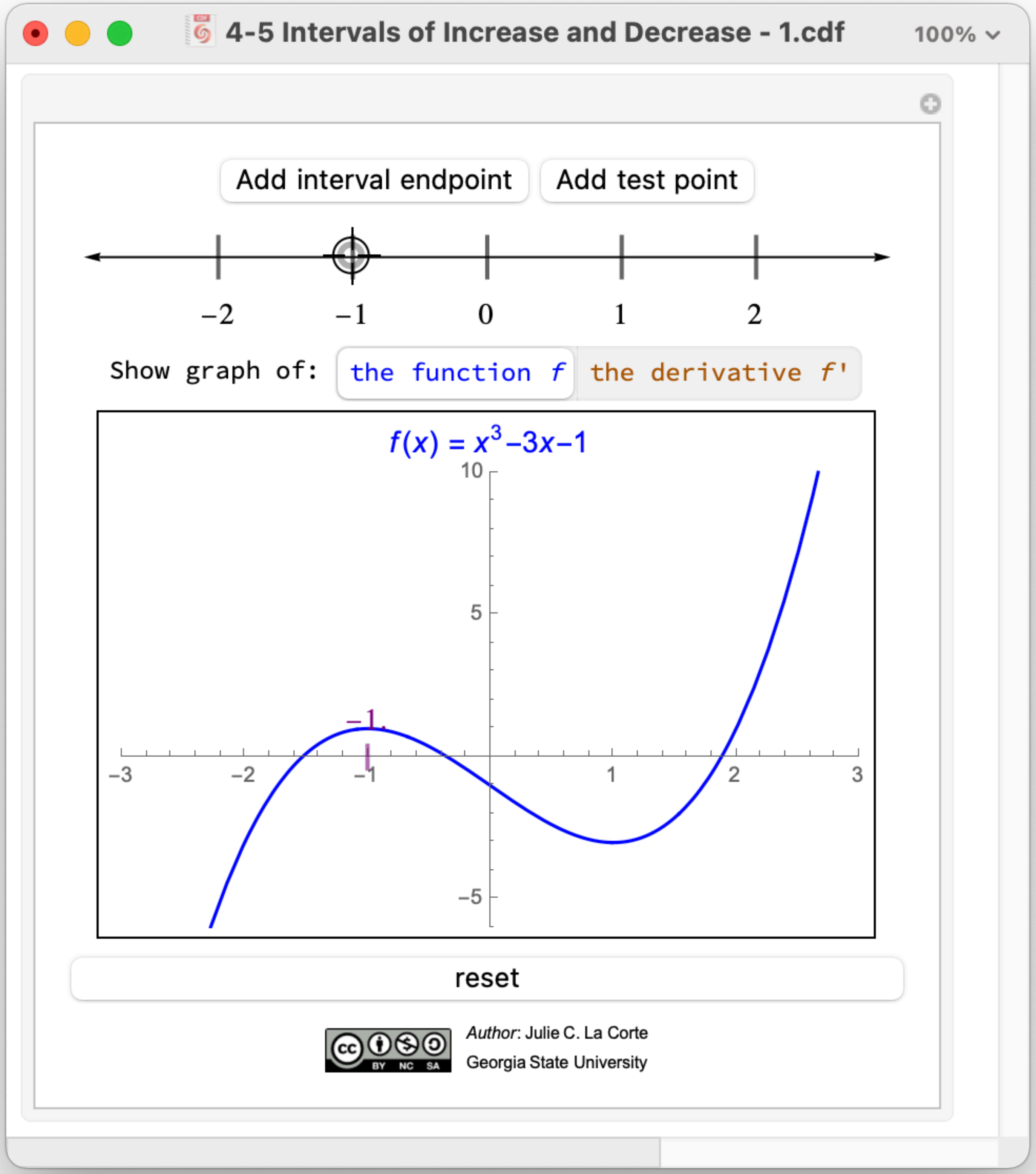
Each added point starts out randomly placed.



Applets

11. Intervals of Increase/Decrease

The crosshairs can then be dragged to wherever the student thinks the added point should be.



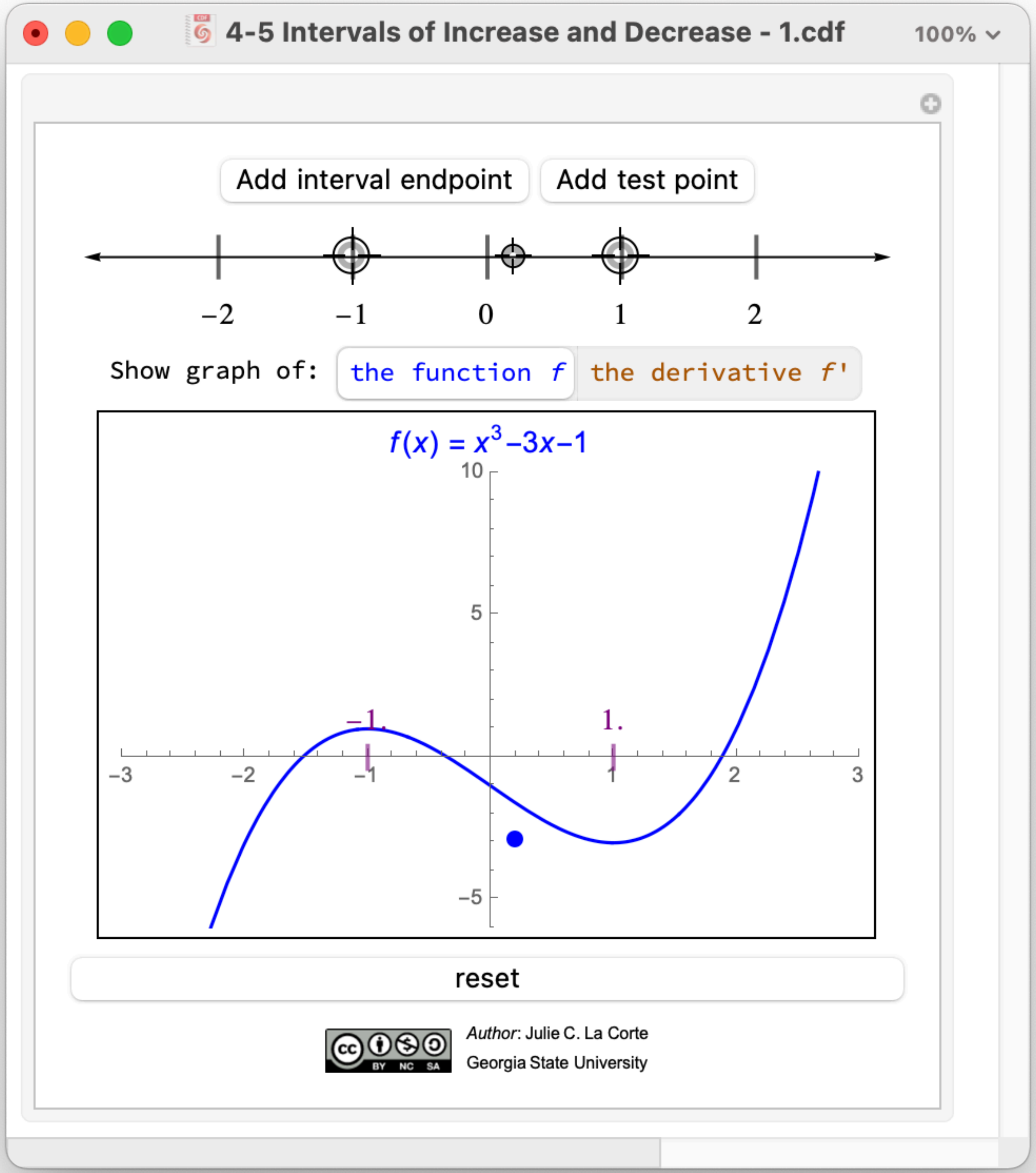
Applets

11. Intervals of Increase/Decrease

When a test point is added, the sign of the derivative is represented by a blue point (–) or a red point (+).

These colored points are secretly points on the graph of the derivative.

I tell the students it's like playing Battleship (if they know what that is).

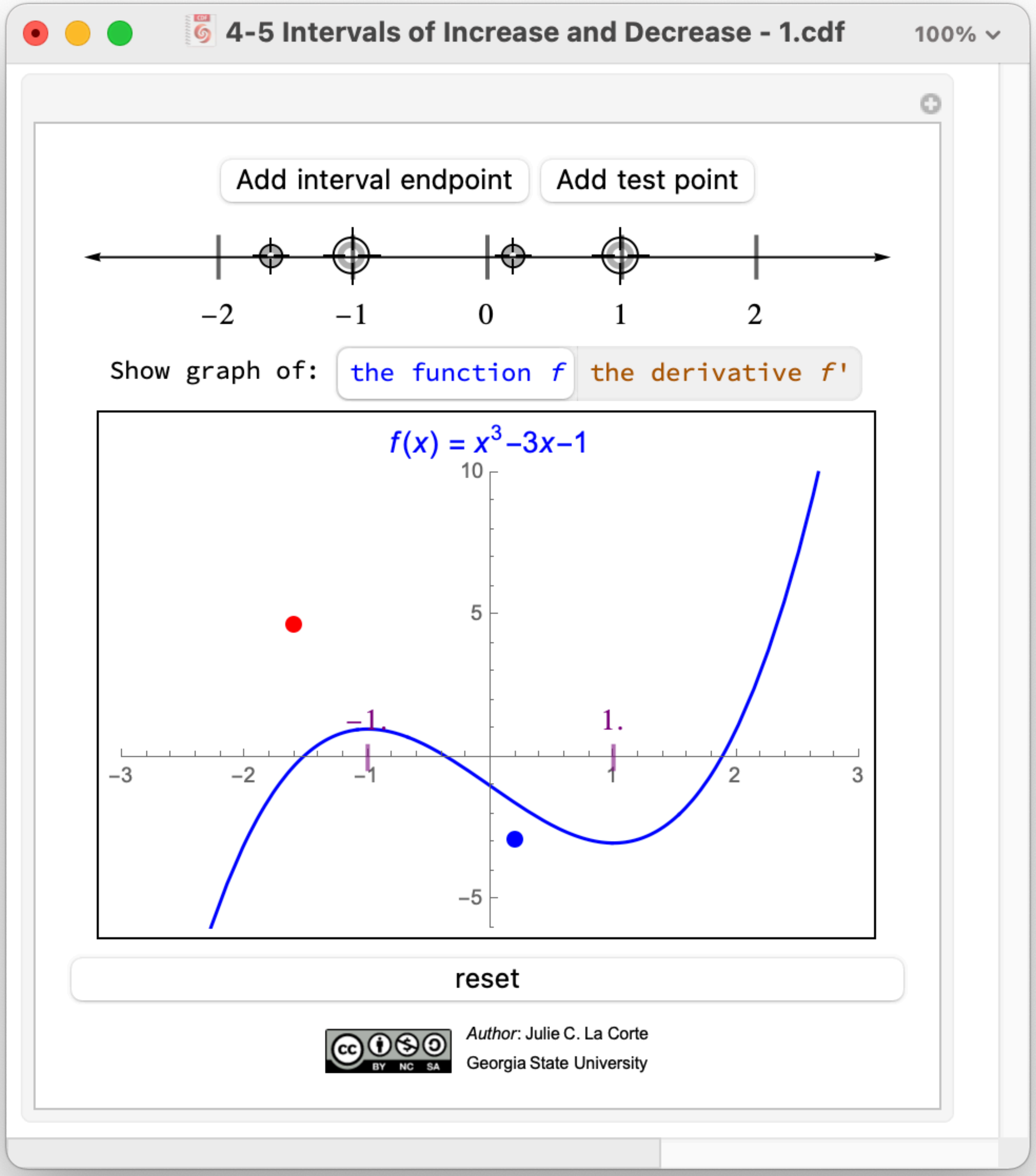


Applets

11. Intervals of Increase/Decrease

The large crosshairs are interval endpoints.

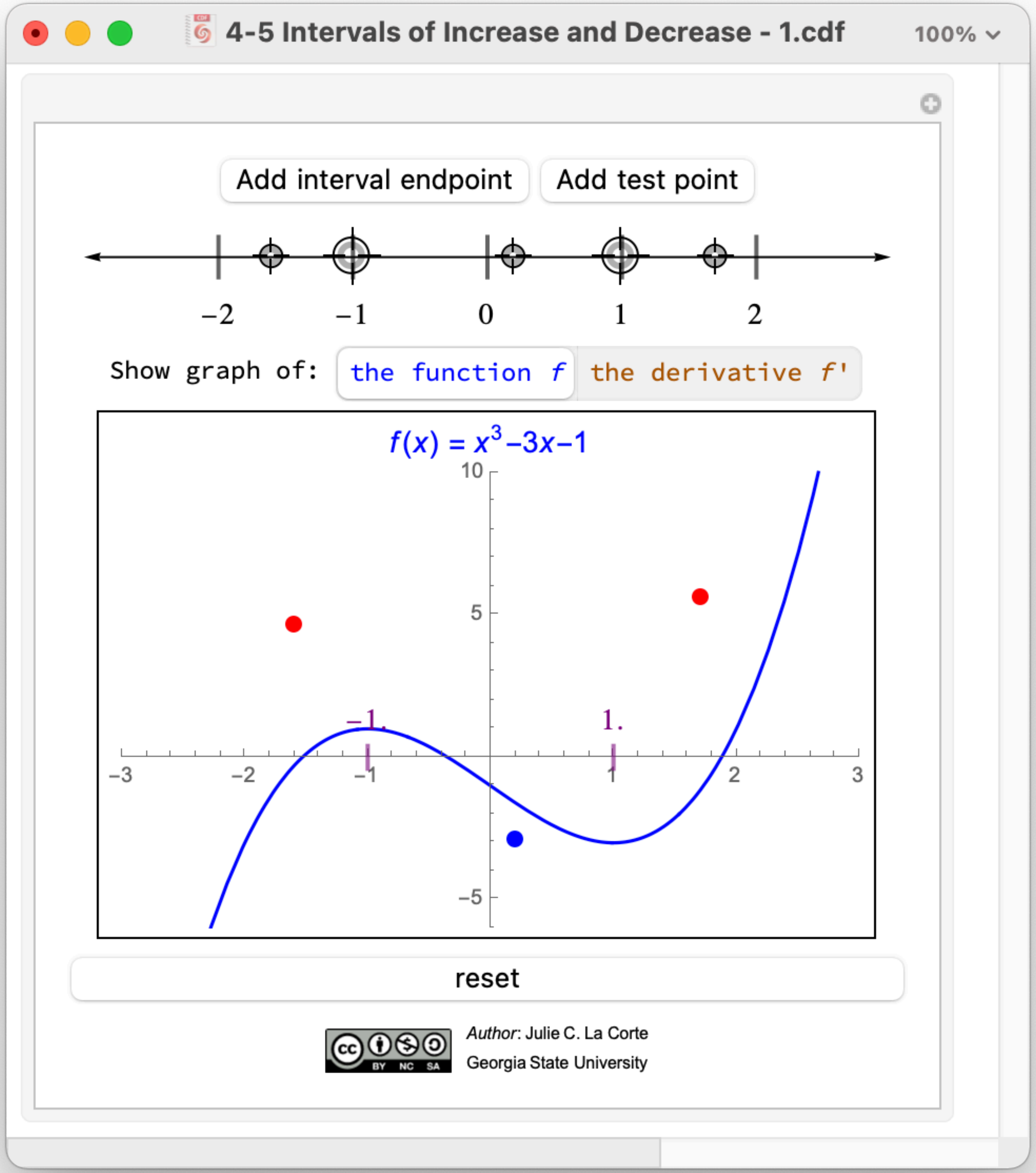
They're represented in the graph by ticks on the x-axis.



Applets

11. Intervals of Increase/Decrease

The small crosshairs are test points.

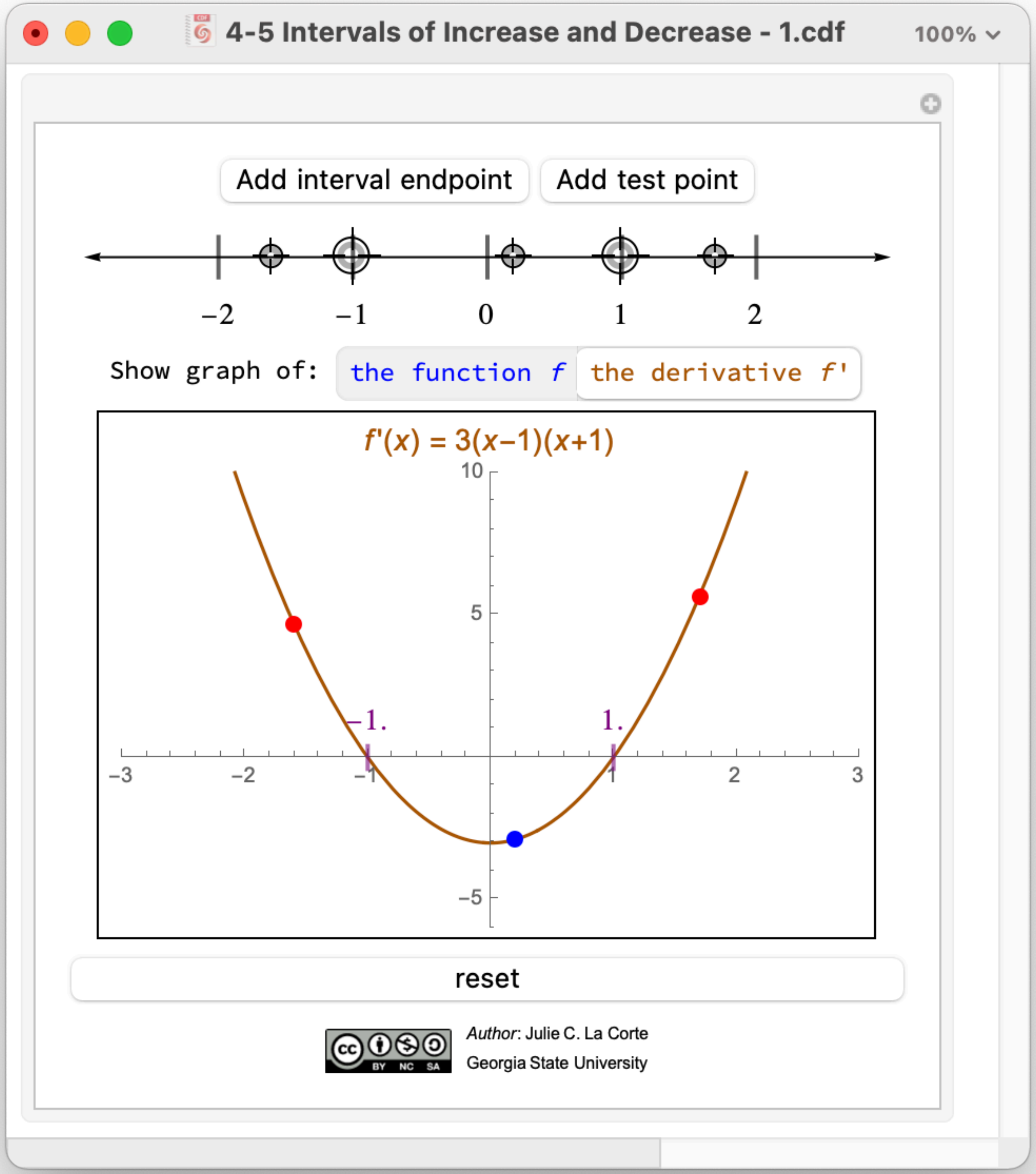


Applets

11. Intervals of Increase/Decrease

Once the problem is completed, the graphs of the function and its derivative can be compared.

In class, working with this applet feels like playing a game —we know what the graph of f looks like from the start, but our goal is to convince someone who *can't* see the graph where f increases and decreases.

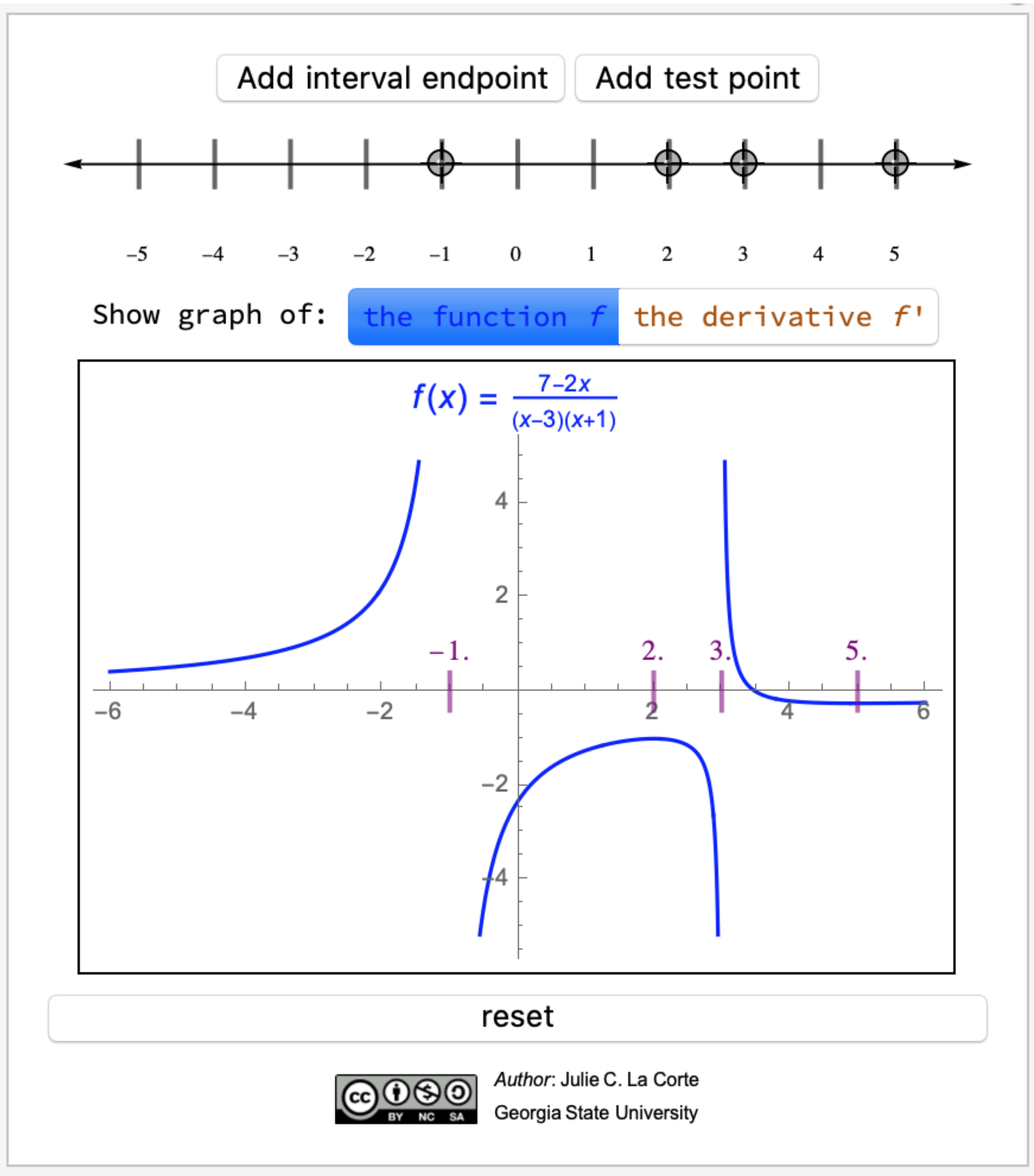


Applets

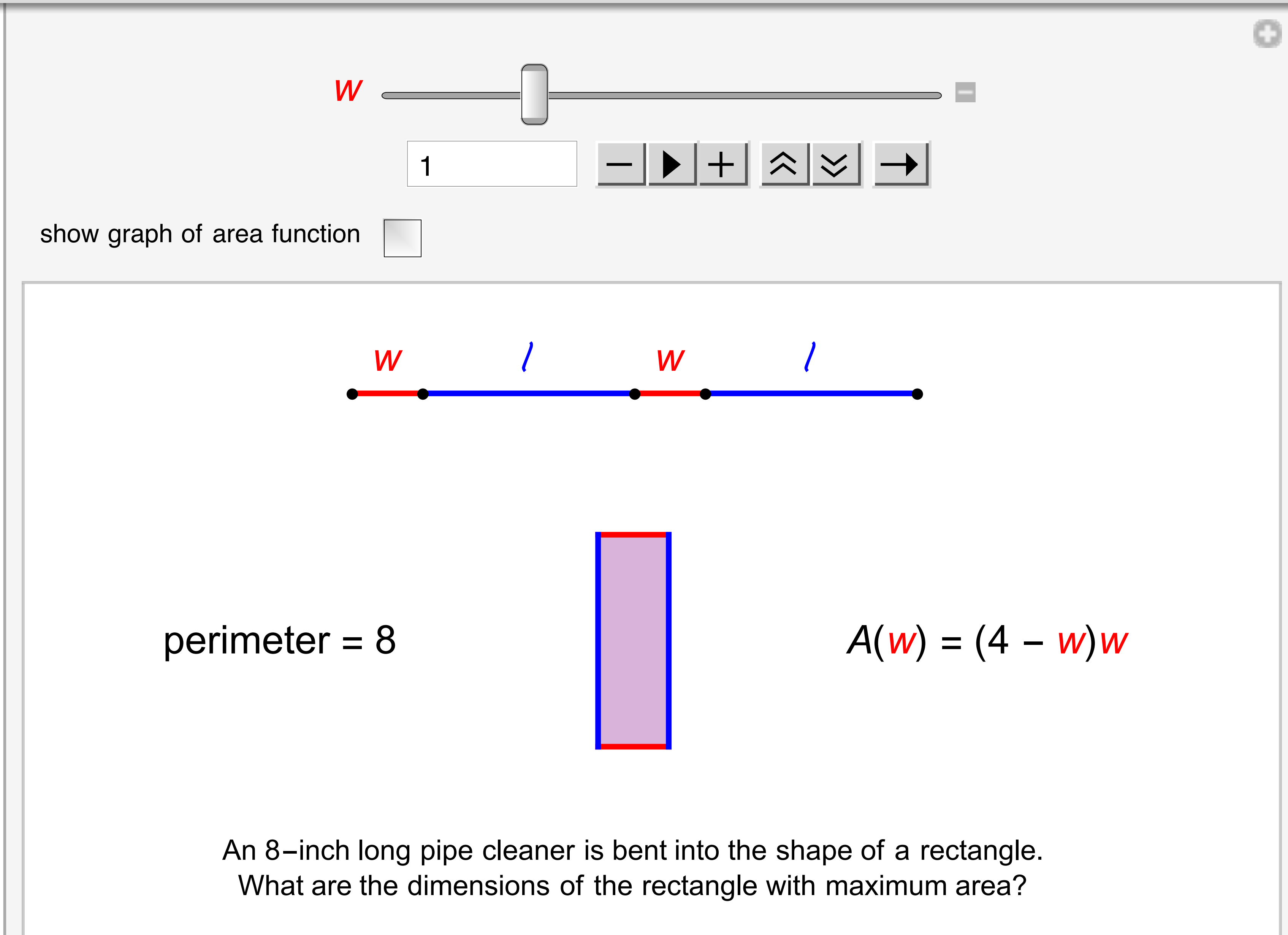
11. Intervals of Increase/Decrease

At this time I have four of these exercises worked up as applets.

It's very easy to change the function for new exercises.



12. Applied Optimization Problems (1 of 3)

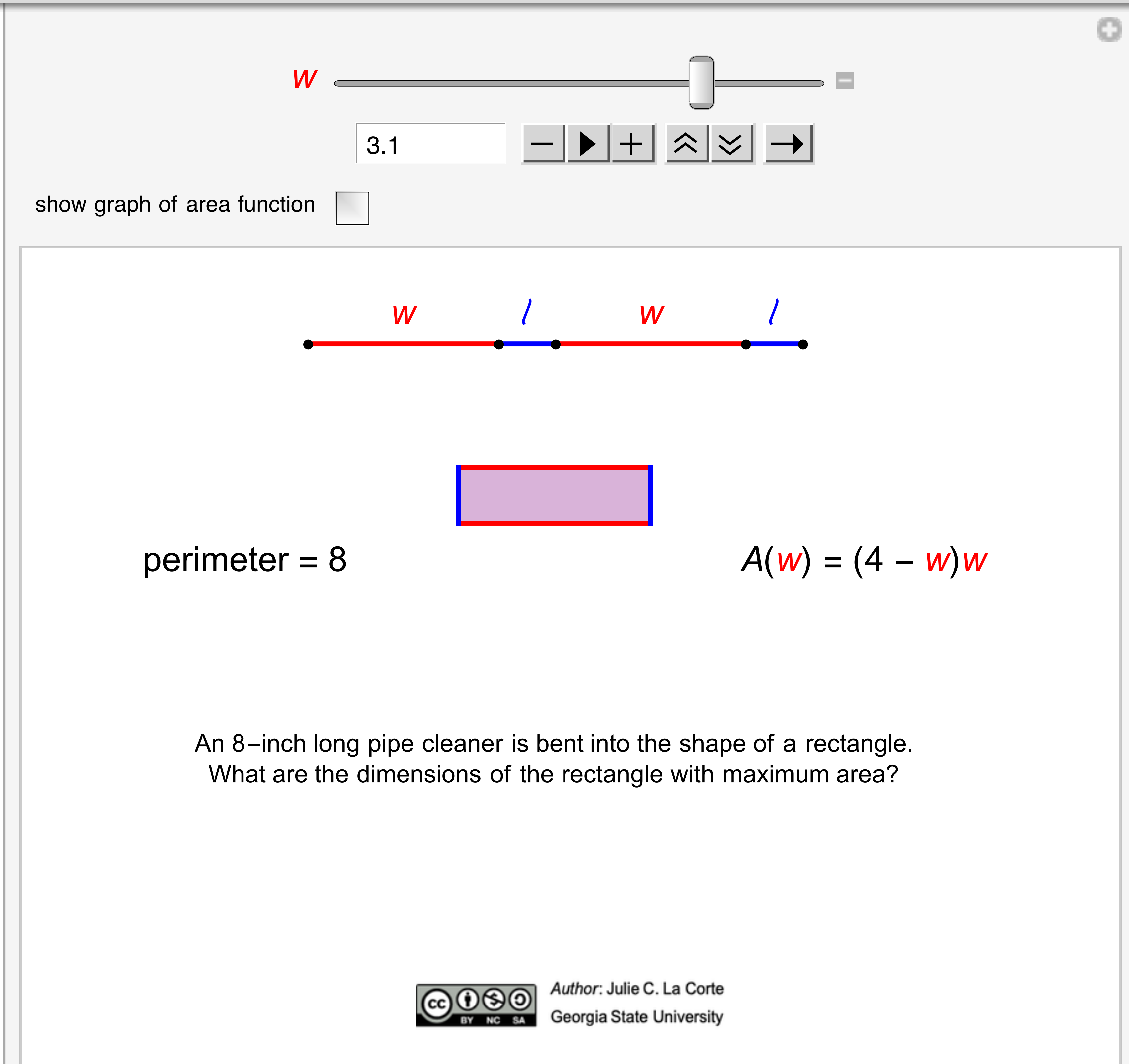


Source for problem: Tricia Van Brunt (Wake Tech CC) and Julia Smith (Wake Tech CC), "Paper, pipe cleaners, and polynomials." 2019 AMATYC Annual Conference: Milwaukee, WI.

Optimization problems in Calculus 1 can all be solved using the same general strategy.

But students may lose sight of the purpose of each step when their process is not grounded in the facts of the problem.

12. Applied Optimization Problems (1 of 3)



In a sequence of three applets, I introduce a general strategy to the students.

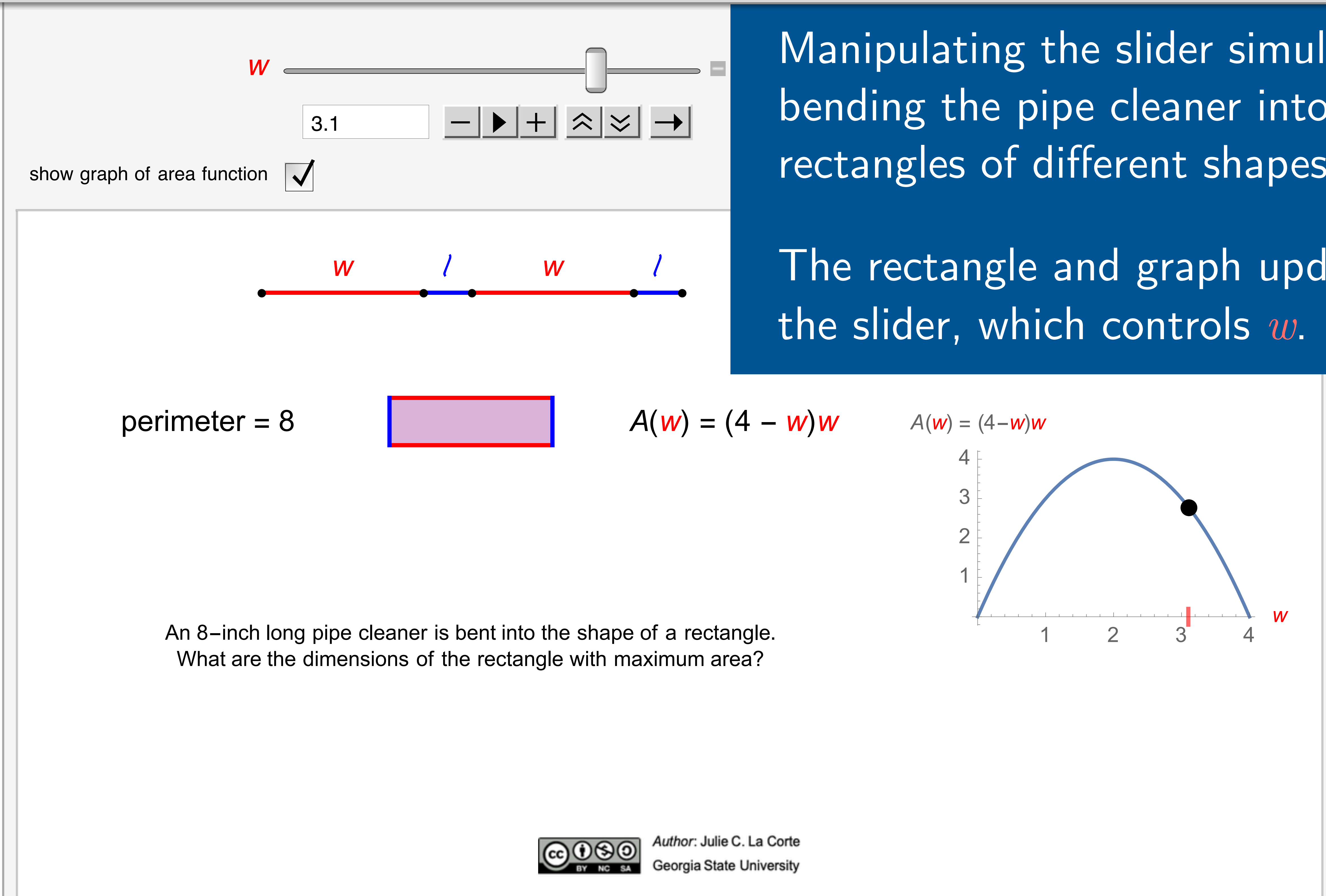
First comes *understanding the problem*.

In this example, the next step is *identifying the quantity to be optimized*.

12. Applied Optimization Problems (1 of 3)

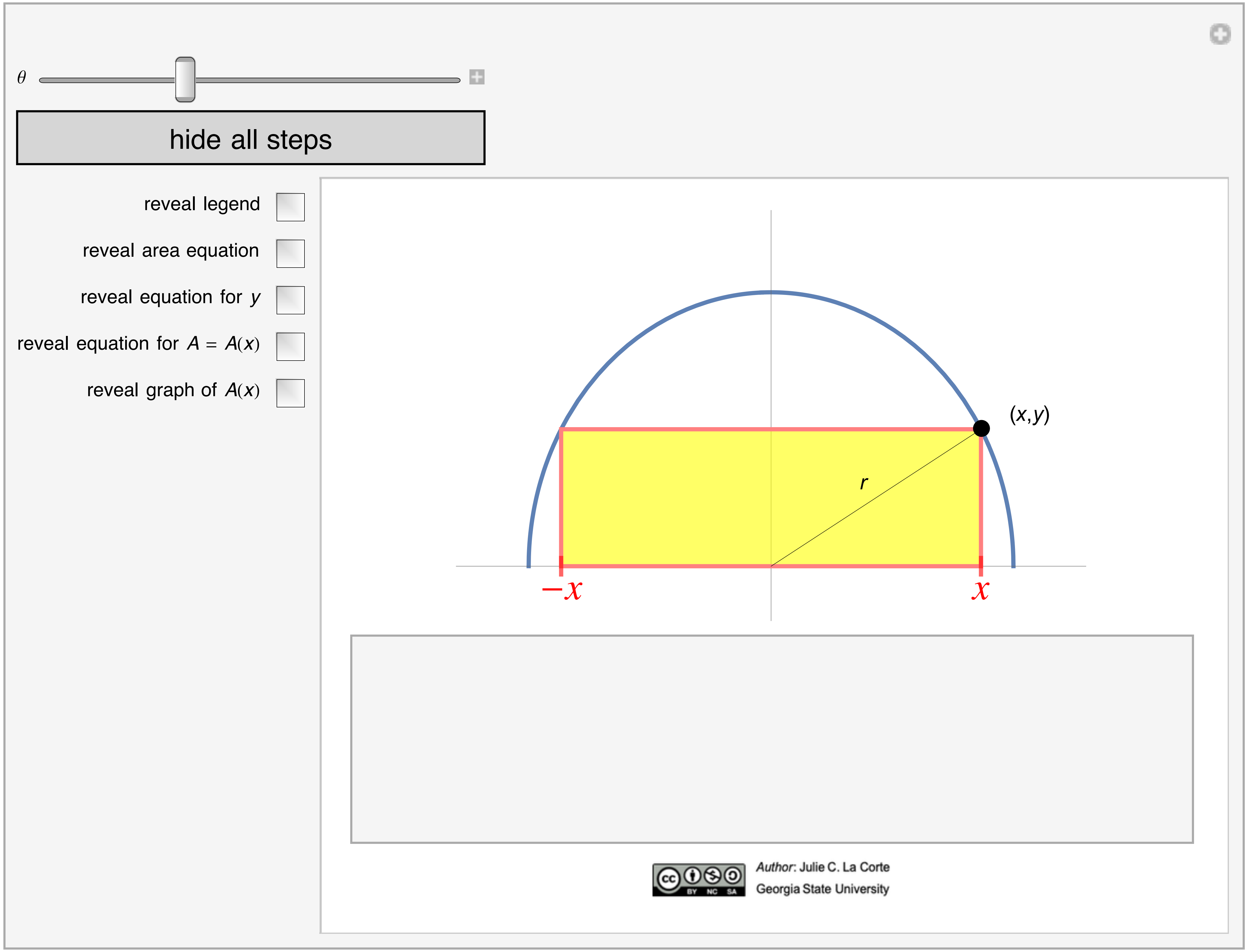
Manipulating the slider simulates bending the pipe cleaner into rectangles of different shapes.

The rectangle and graph update with the slider, which controls w.



12. Applied Optimization Problems (2 of 3)

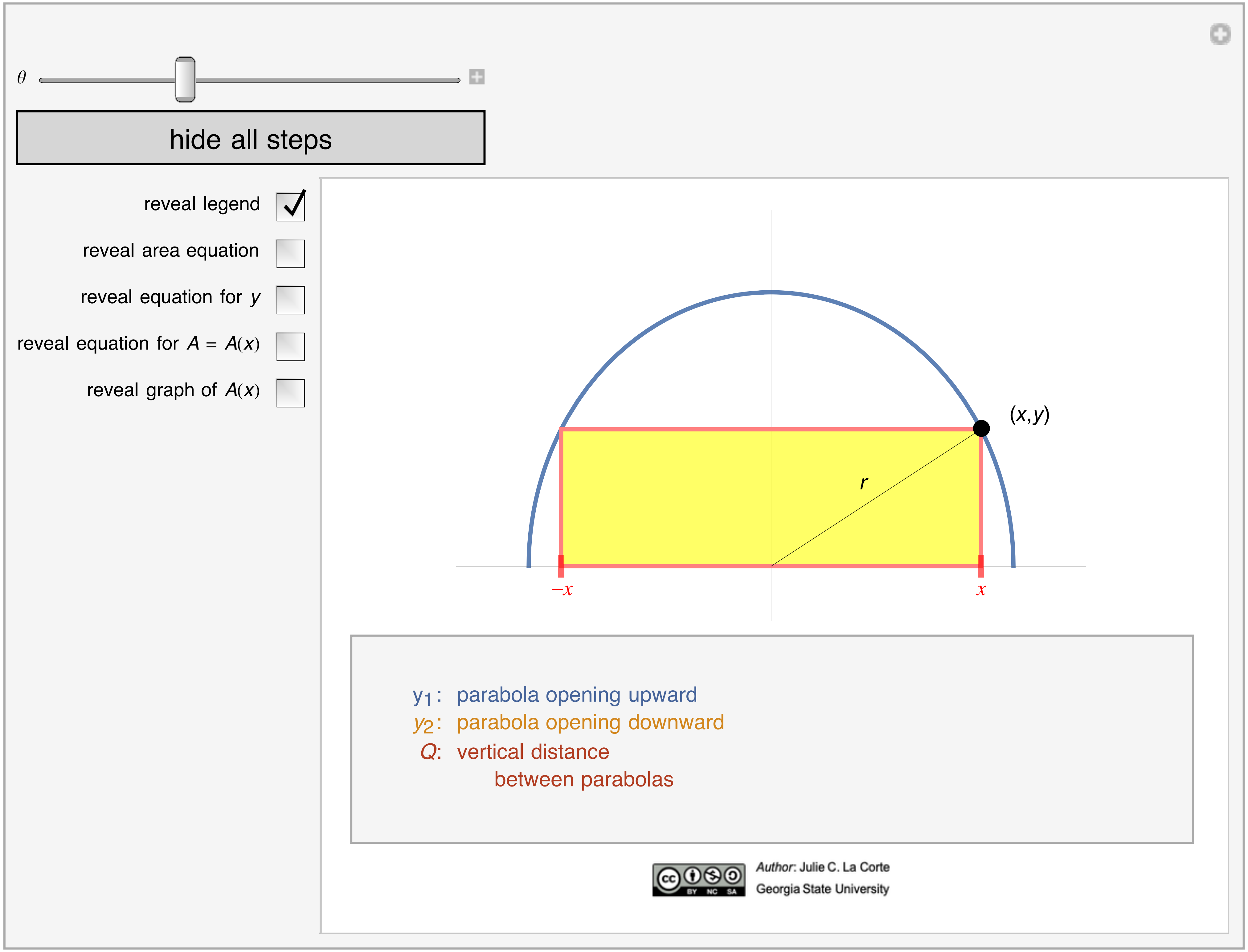
The general strategy given in the Workbook for these problems includes *writing a legend*.



12. Applied Optimization Problems (2 of 3)

This applet can be used for group work.

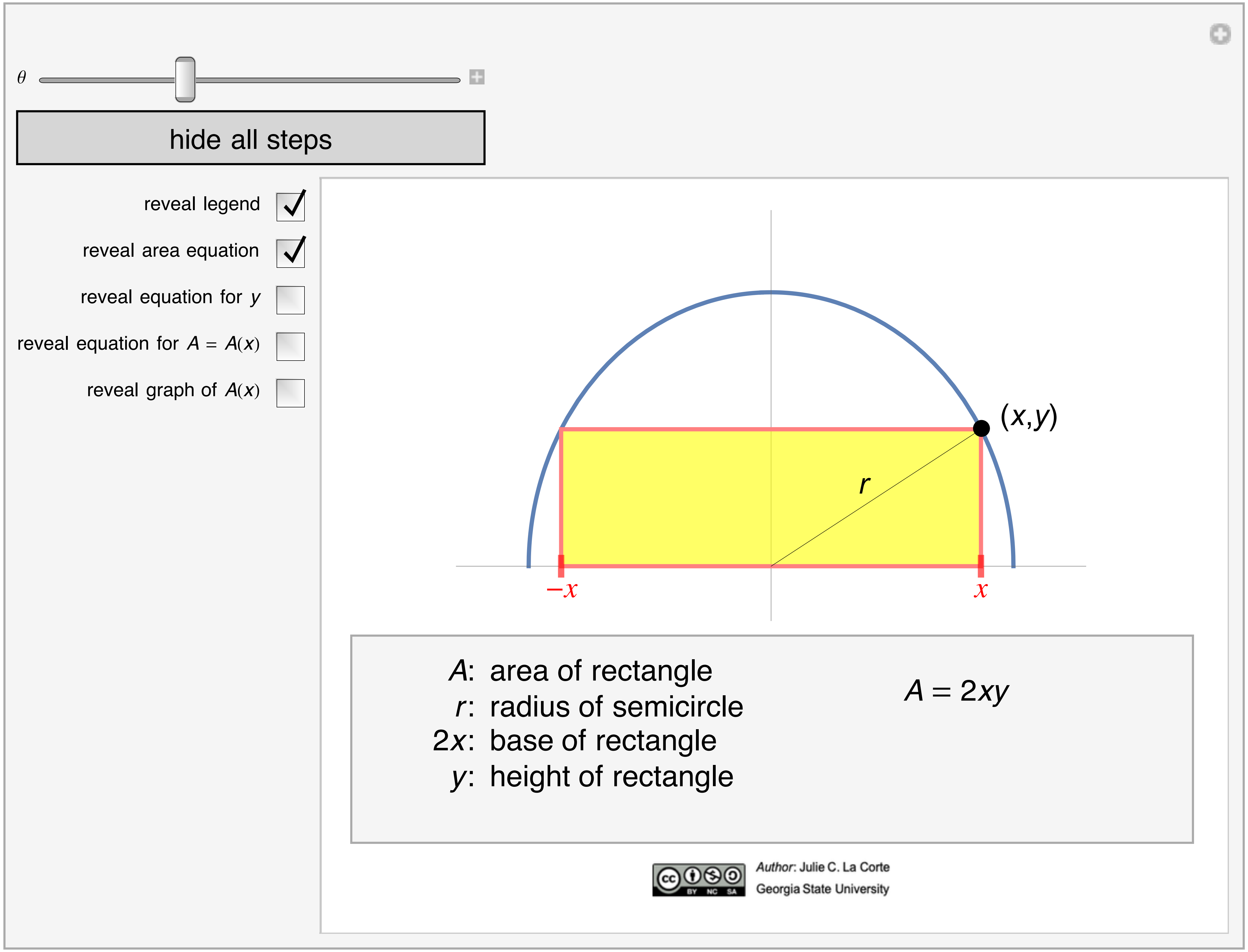
Once the "legend" is revealed, all students can develop their own solutions with the same notation.



Applets

12. Applied Optimization Problems (2 of 3)

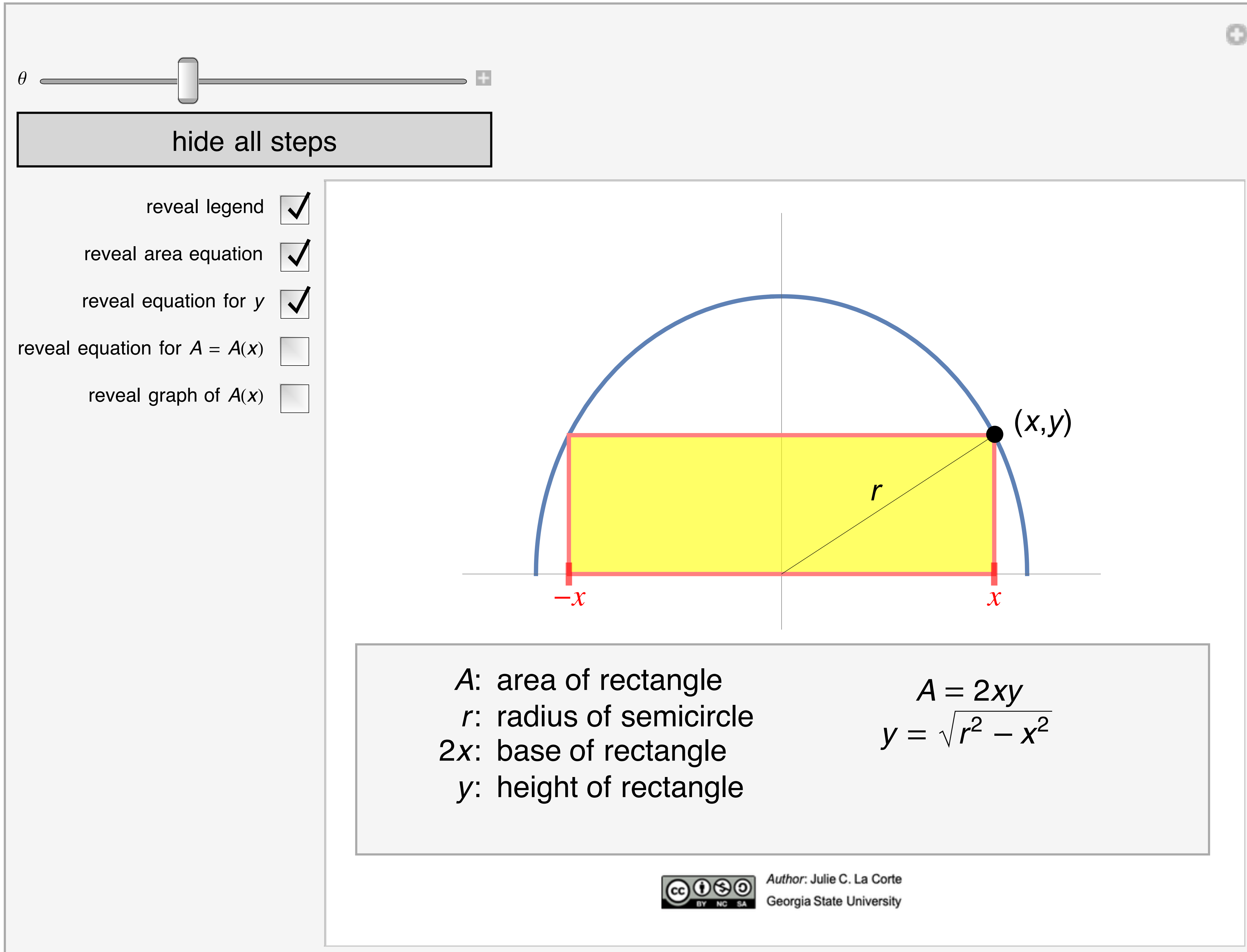
As with other types of multipart problems, the applet hides the technical details and emphasizes the “Big Picture” strategy.



Applets

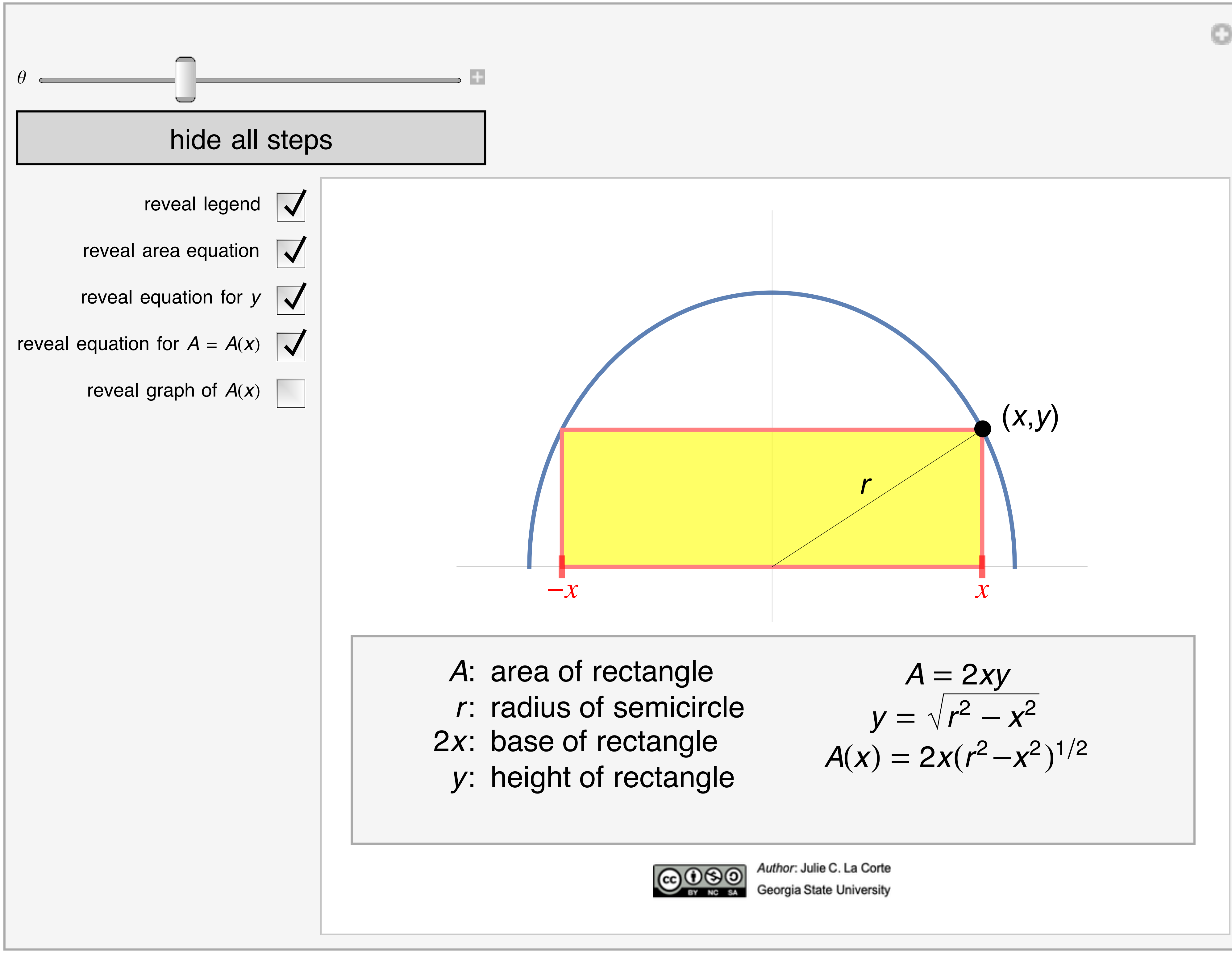
12. Applied Optimization Problems (2 of 3)

The applets enable kinesthetic learners to ground their intuition in physical interaction with a live working model.



12. Applied Optimization Problems (2 of 3)

Even if a student gets stuck on a step along the way, they can still use the graph (*next two slides*) to find an approximate solution.



12. Applied Optimization Problems (2 of 3)

θ

hide all steps

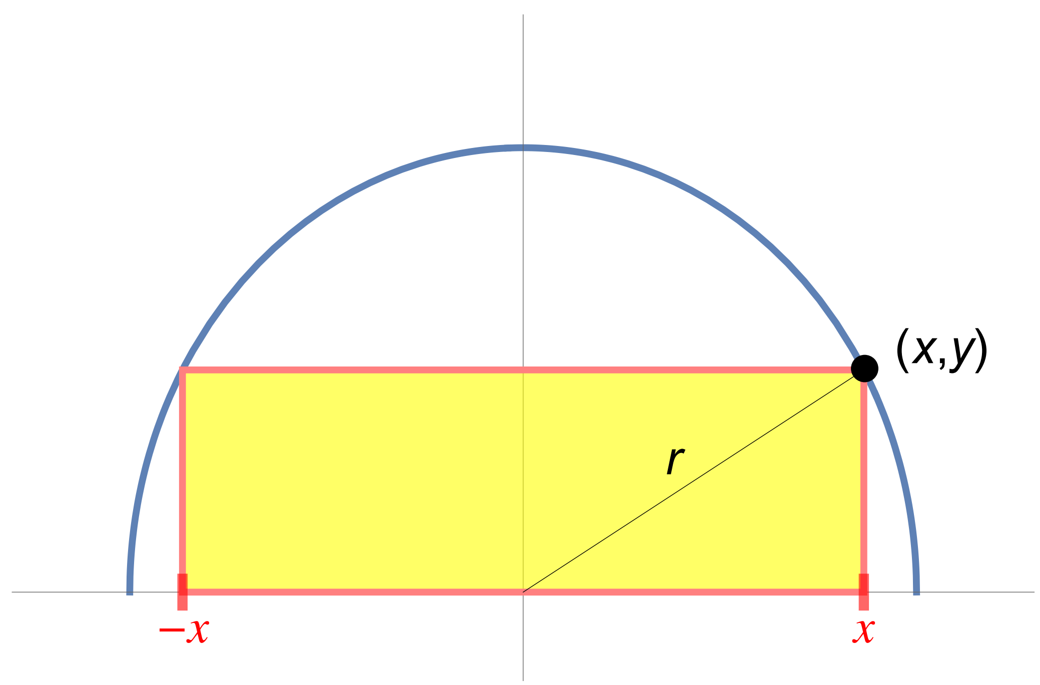
reveal legend ☑

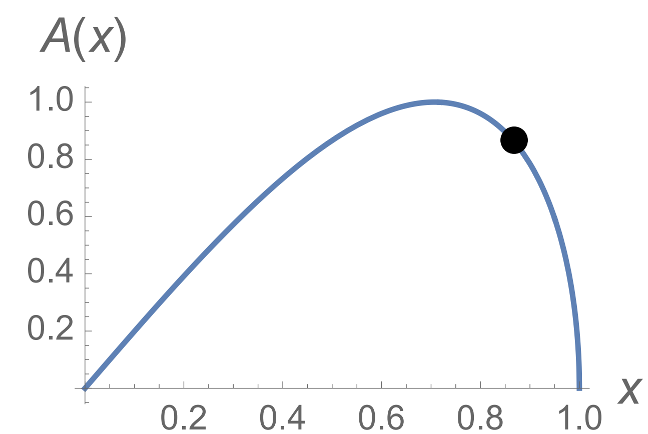
reveal area equation ☑

reveal equation for y ☑

reveal equation for $A = A(x)$ ☑

reveal graph of $A(x)$ ☑





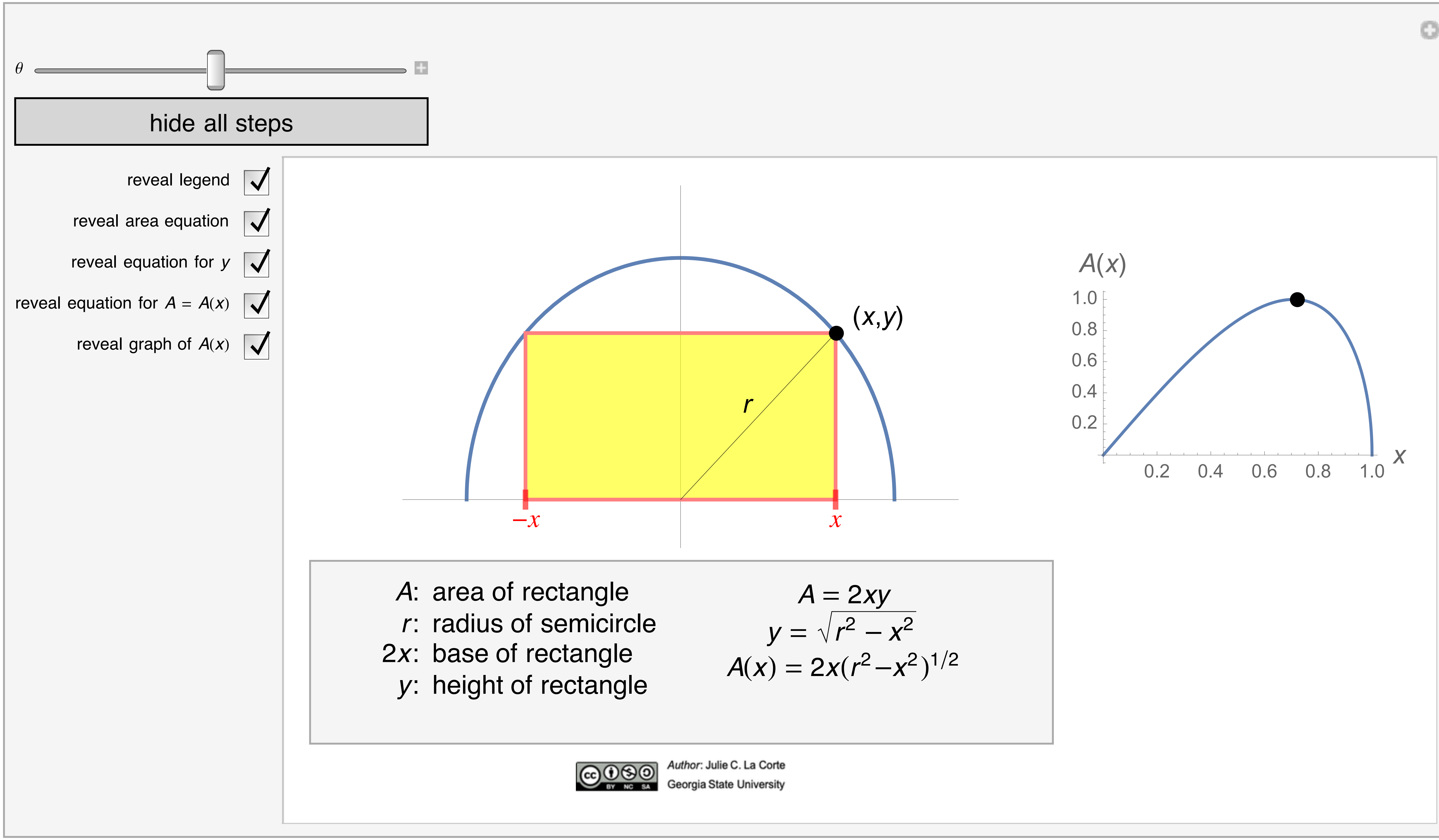
A: area of rectangle
r: radius of semicircle
$2x$: base of rectangle
y: height of rectangle

$$A = 2xy$$
$$y = \sqrt{r^2 - x^2}$$
$$A(x) = 2x(r^2 - x^2)^{1/2}$$

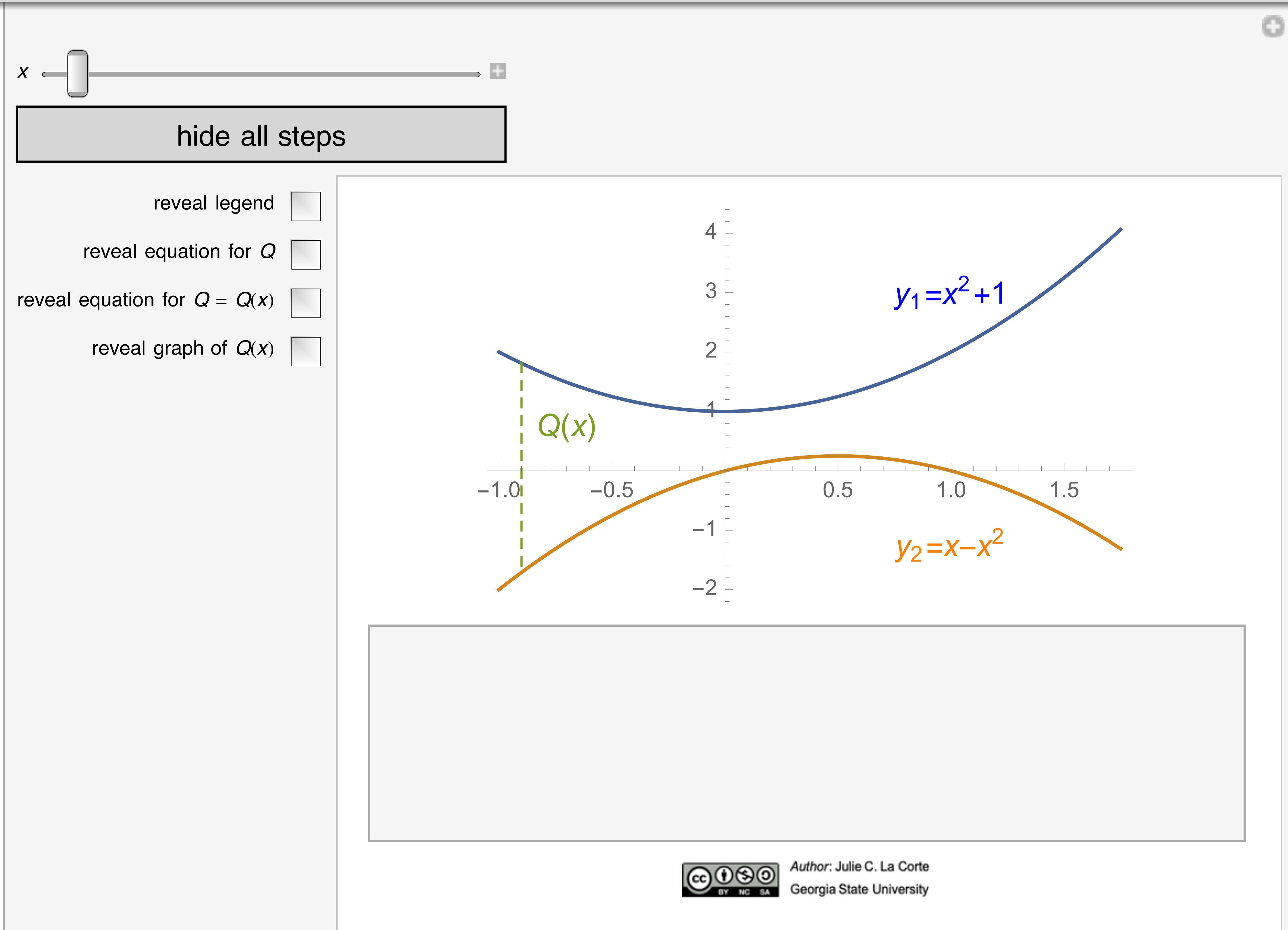
Author: Julie C. La Corte
Georgia State University

Applets

12. Applied Optimization Problems (2 of 3)



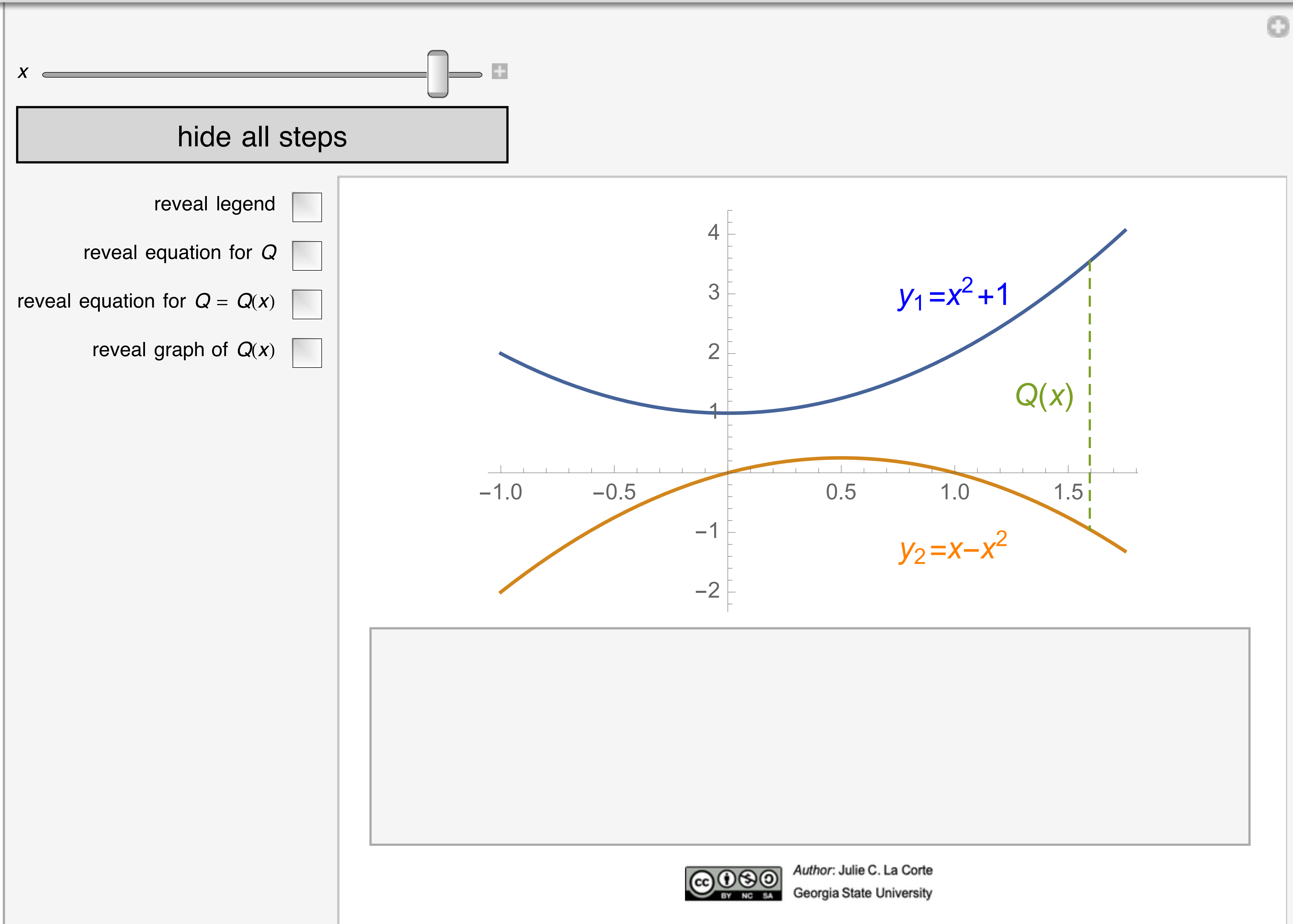
12. Applied Optimization Problems (3 of 3)



"The vertical distance between two graphs" is concretely identifiable in the picture.

Applets

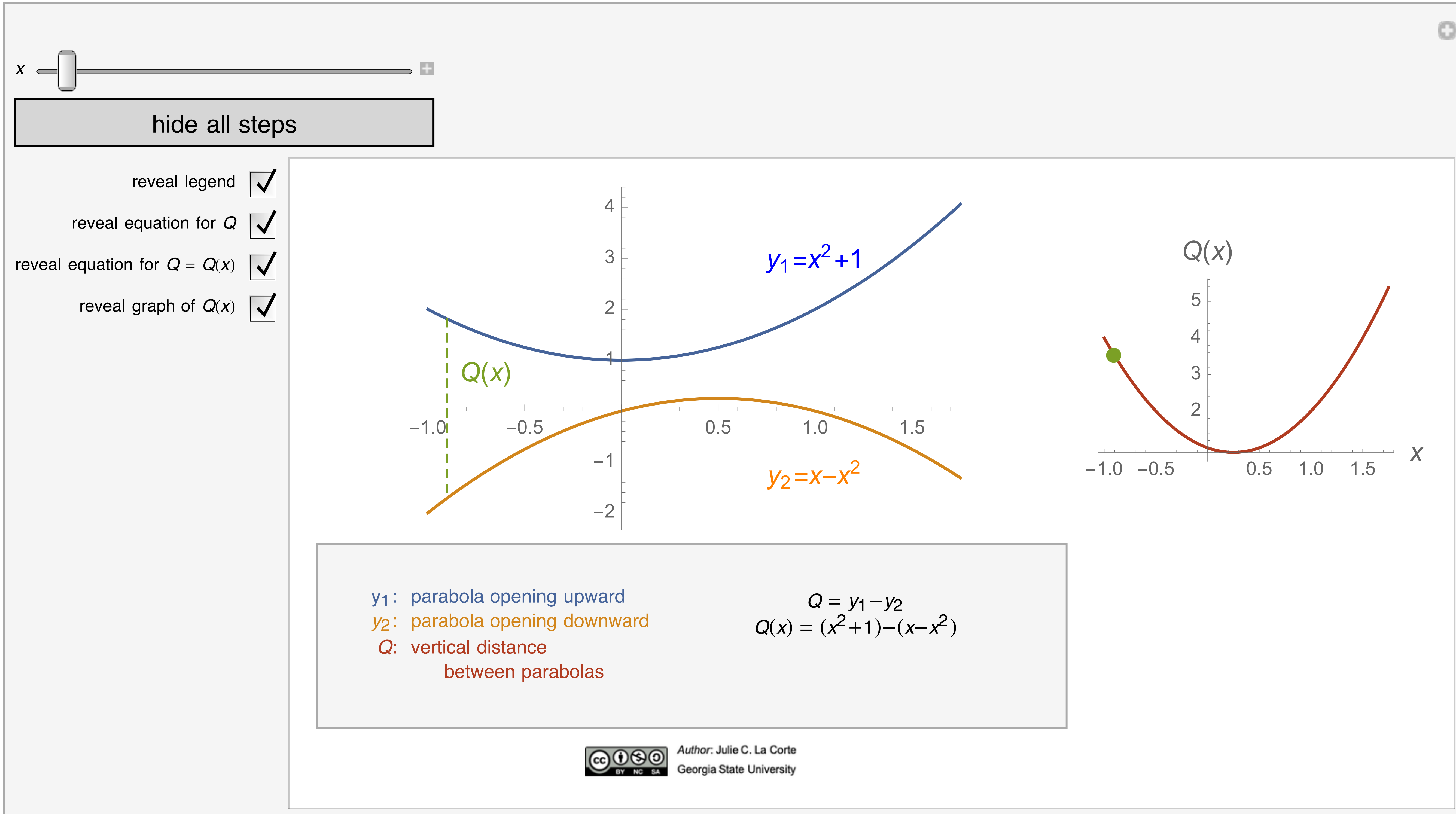
12. Applied Optimization Problems (3 of 3)



The fact that this vertical distance is a function of x is easy to see when we use the slider to move x.

Applets

12. Applied Optimization Problems (3 of 3)



Applets

Curriculum coverage

About half of the Workbook Lessons, each of which covers one textbook section, currently have applets associated with them, some of which were written by other authors.

The other authors' Mathematica applets can be downloaded at the **Wolfram Demonstration Project** website.

- Applets uploaded to Wolfram tend not to be taken down once posted

Beware of linking to web resources that may not be available in the future.

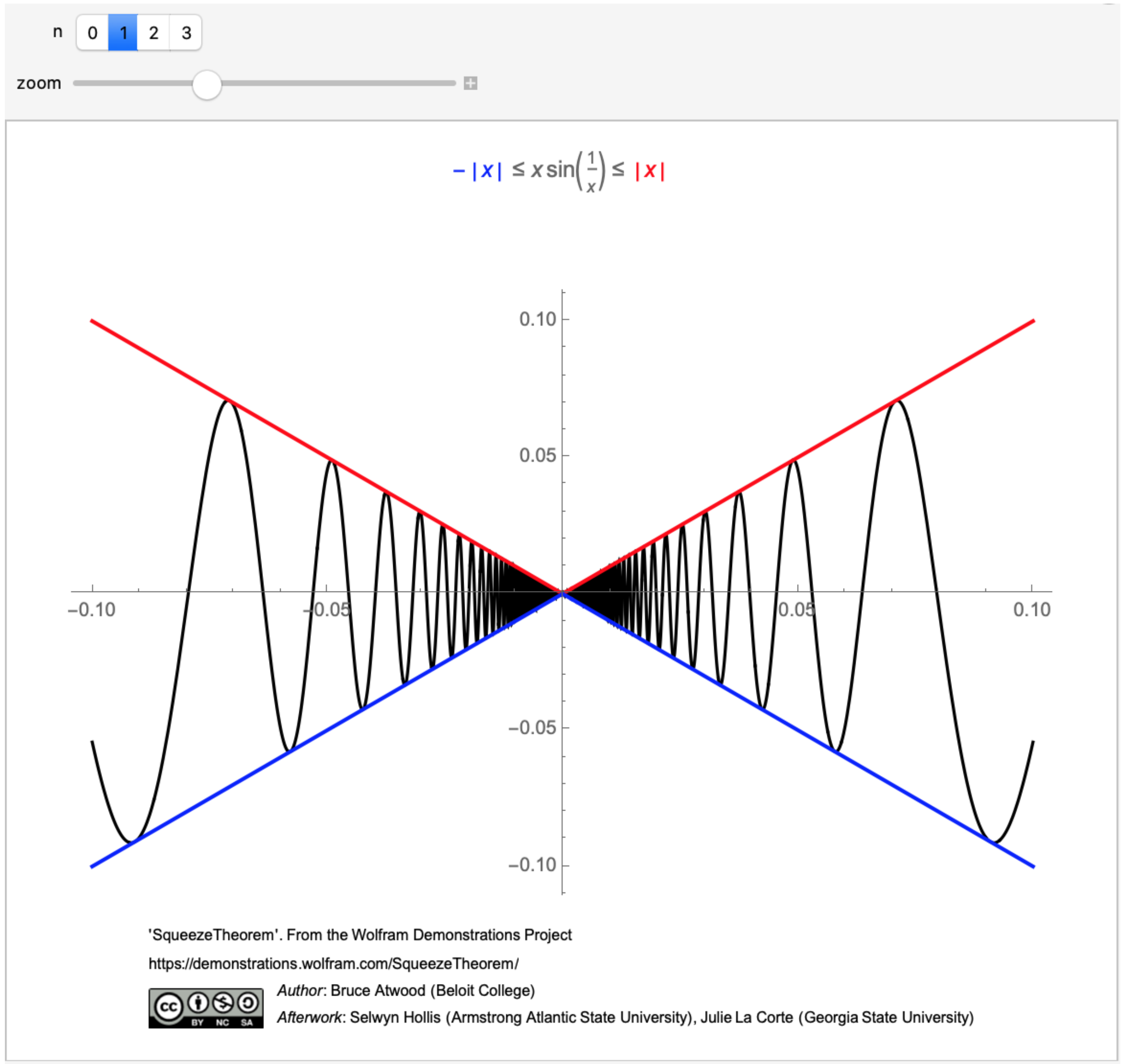
- Dead links in lesson plans are a compelling reason to advocate for accessible institutional repositories for open source teaching materials

Applets

Remixes of other authors' applets

§2.3

- Squeeze Theorem (*Author*: Bruce Atwood, Beloit College, *Remix*: Julie C. La Corte)

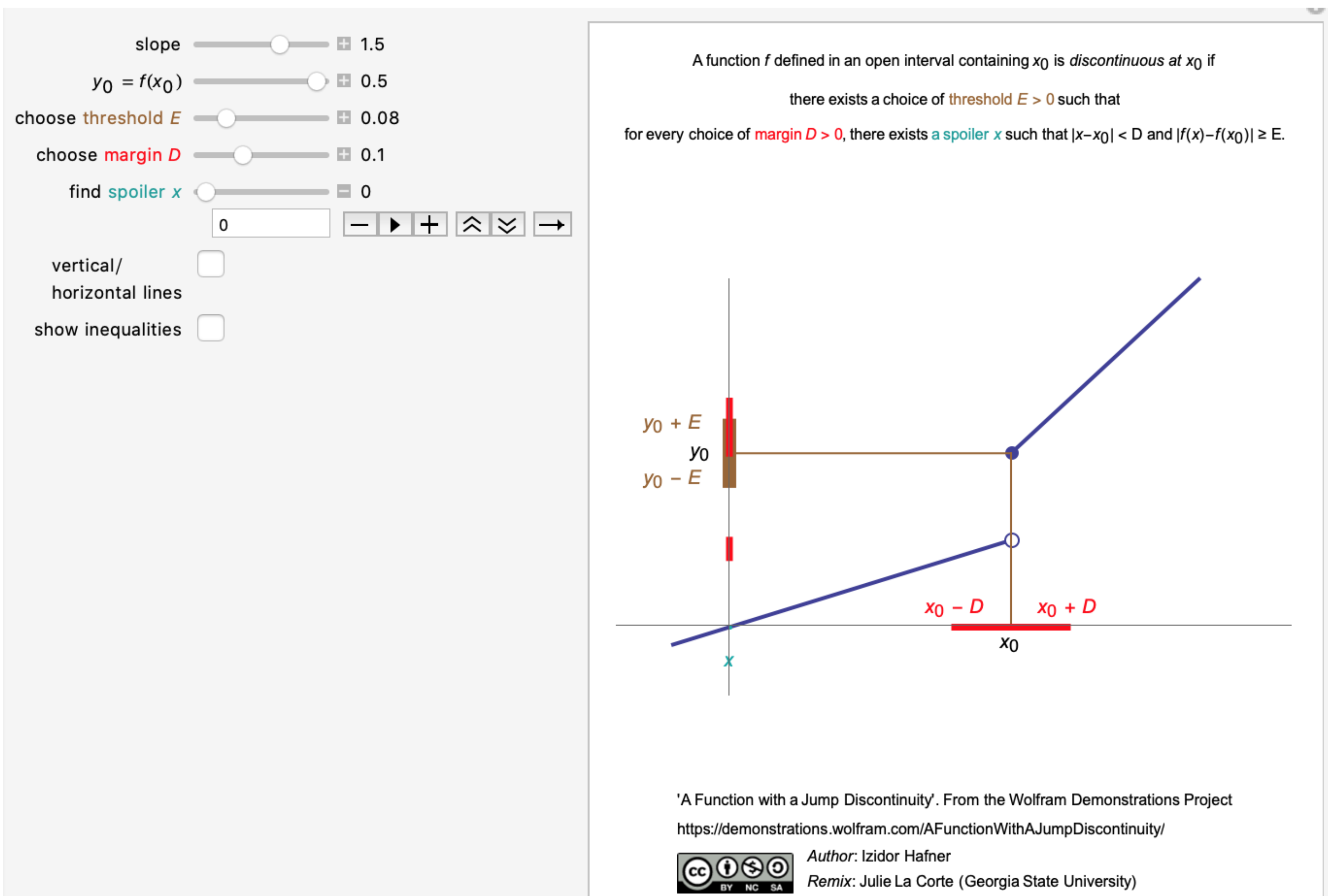


Applets

Remixes of other authors' applets

§2.4

- Formal meaning of discontinuity (*Author*: Izidor Hafner, *Remix*: Julie C. La Corte)

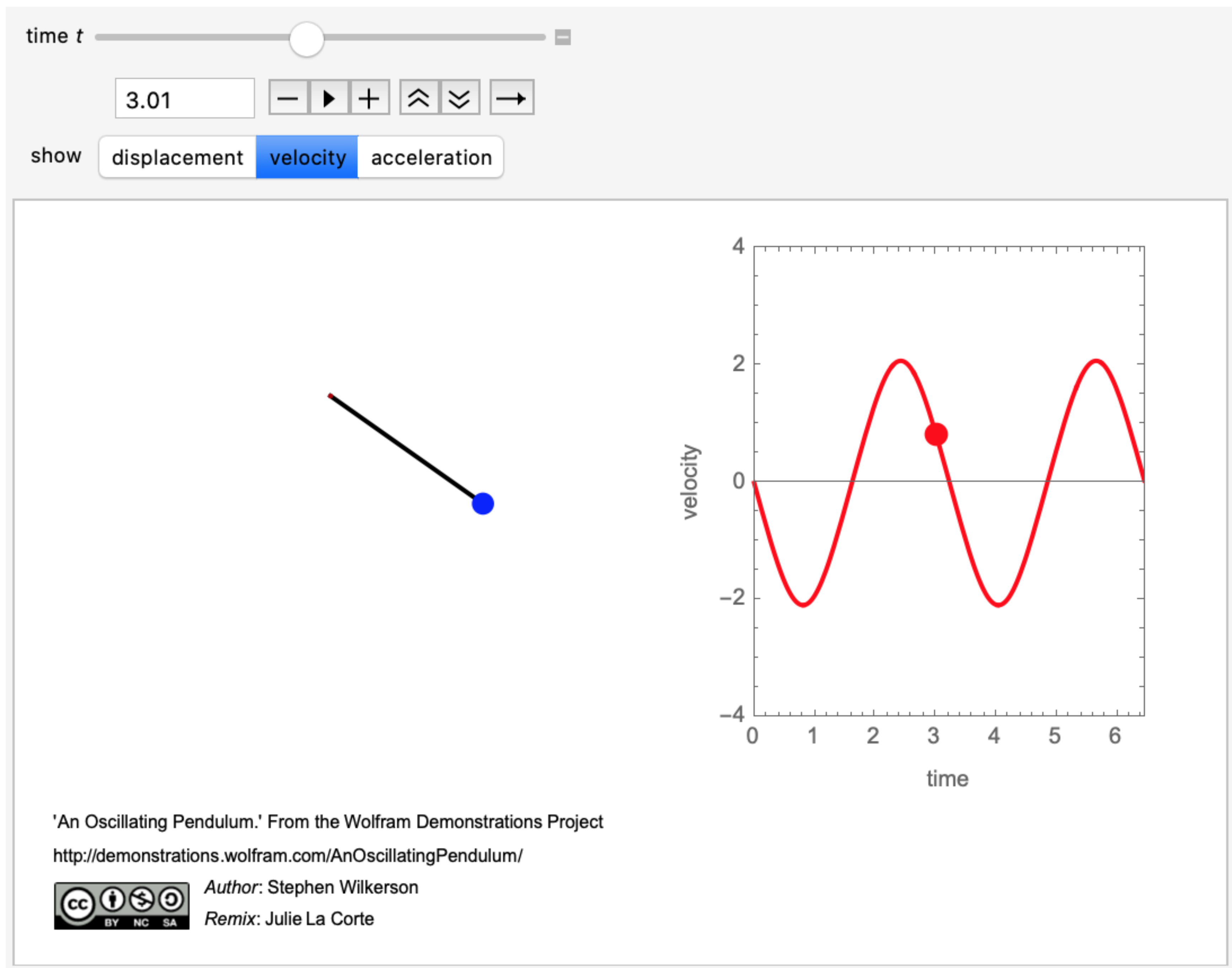


Applets

Remixes of other authors' applets

§3.1

- Pendulum

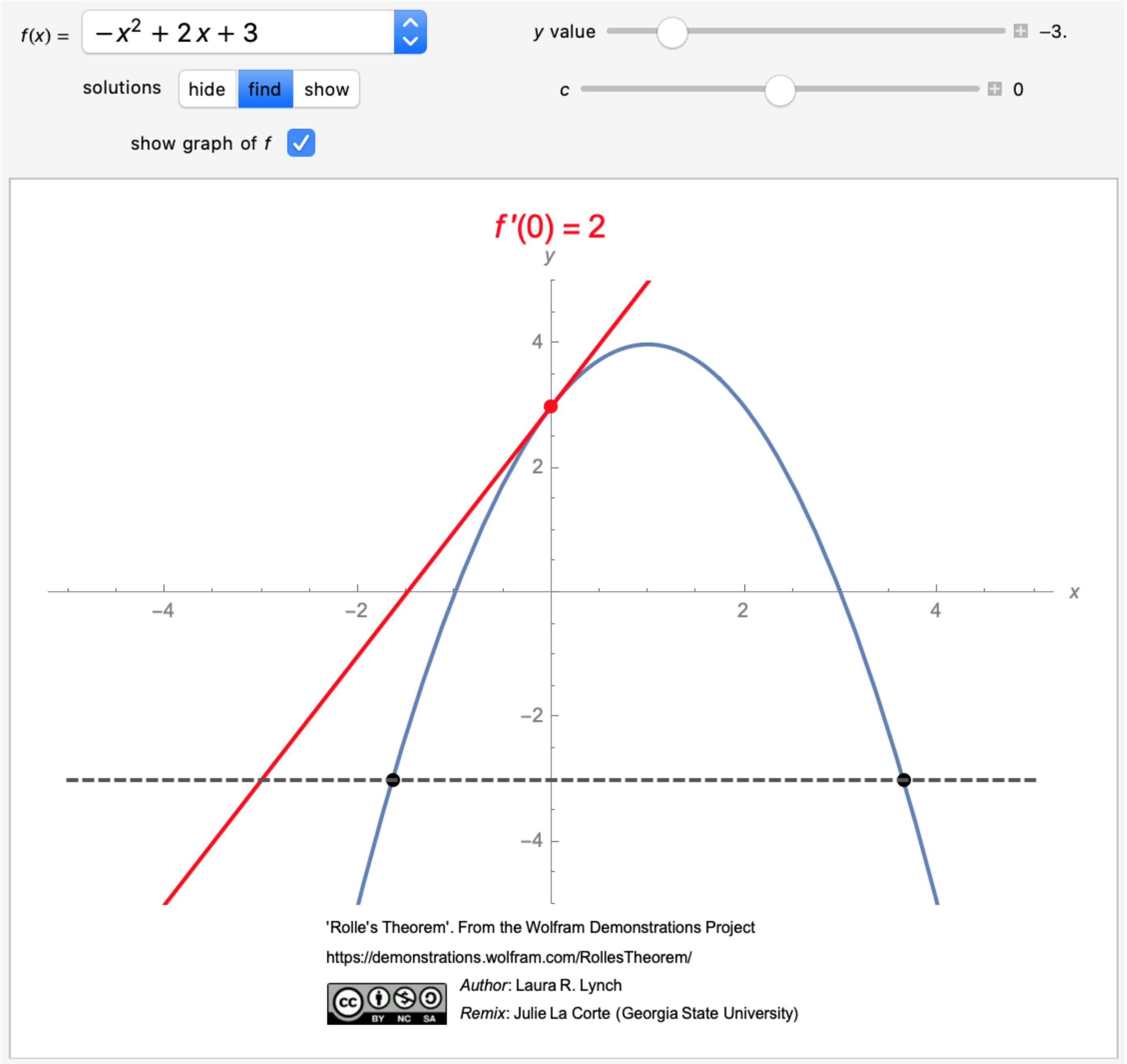


Applets

Remixes of other authors’ applets

§4.4

- Rolle’s Theorem (*Author*: Laura R. Lynch, *Remix*: Julie C. La Corte)

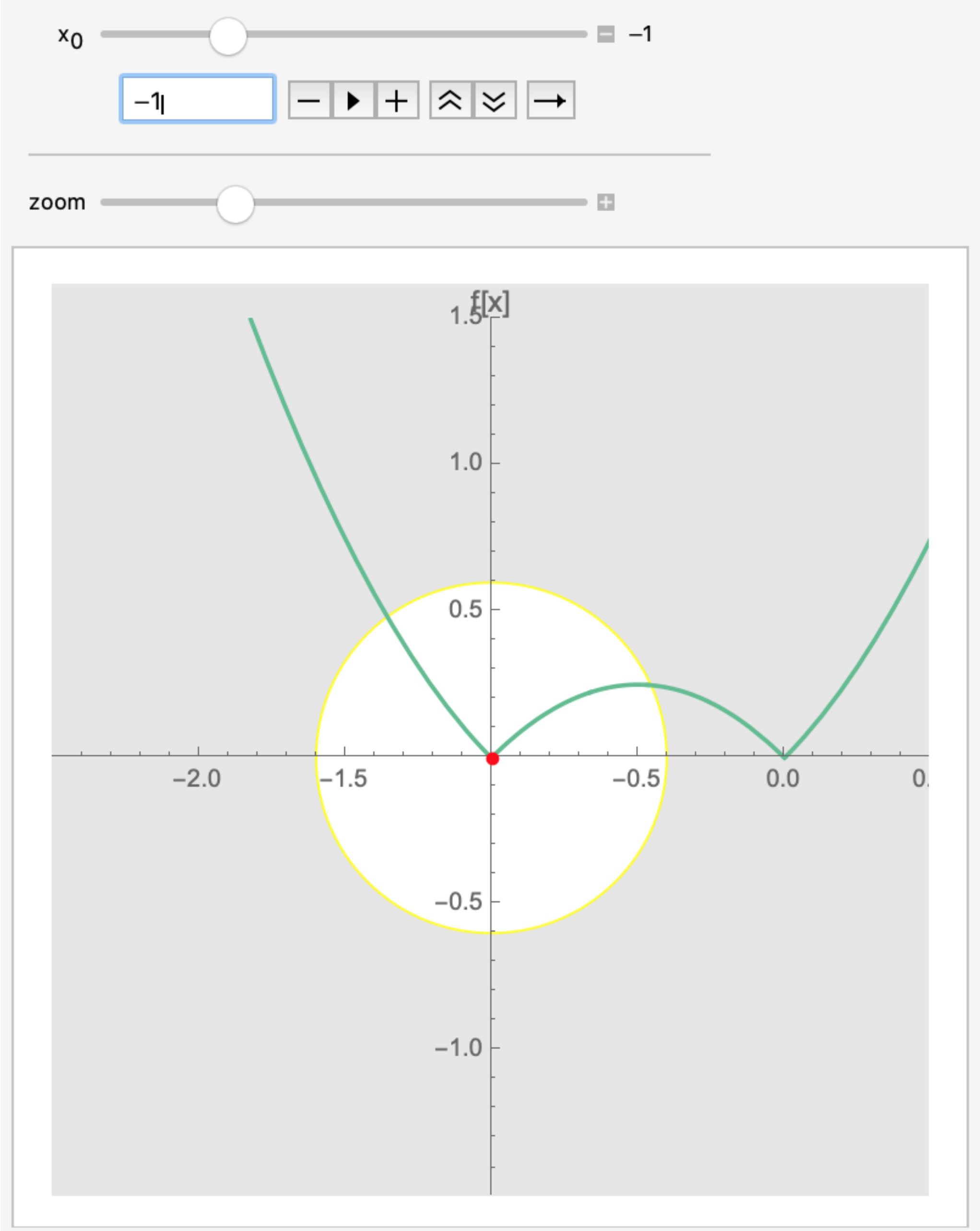


Applets

Other authors' applets

§3.1

- Differentiation microscope (*Authors*: Wolfgang Narrath and Reinhard Simonovits)



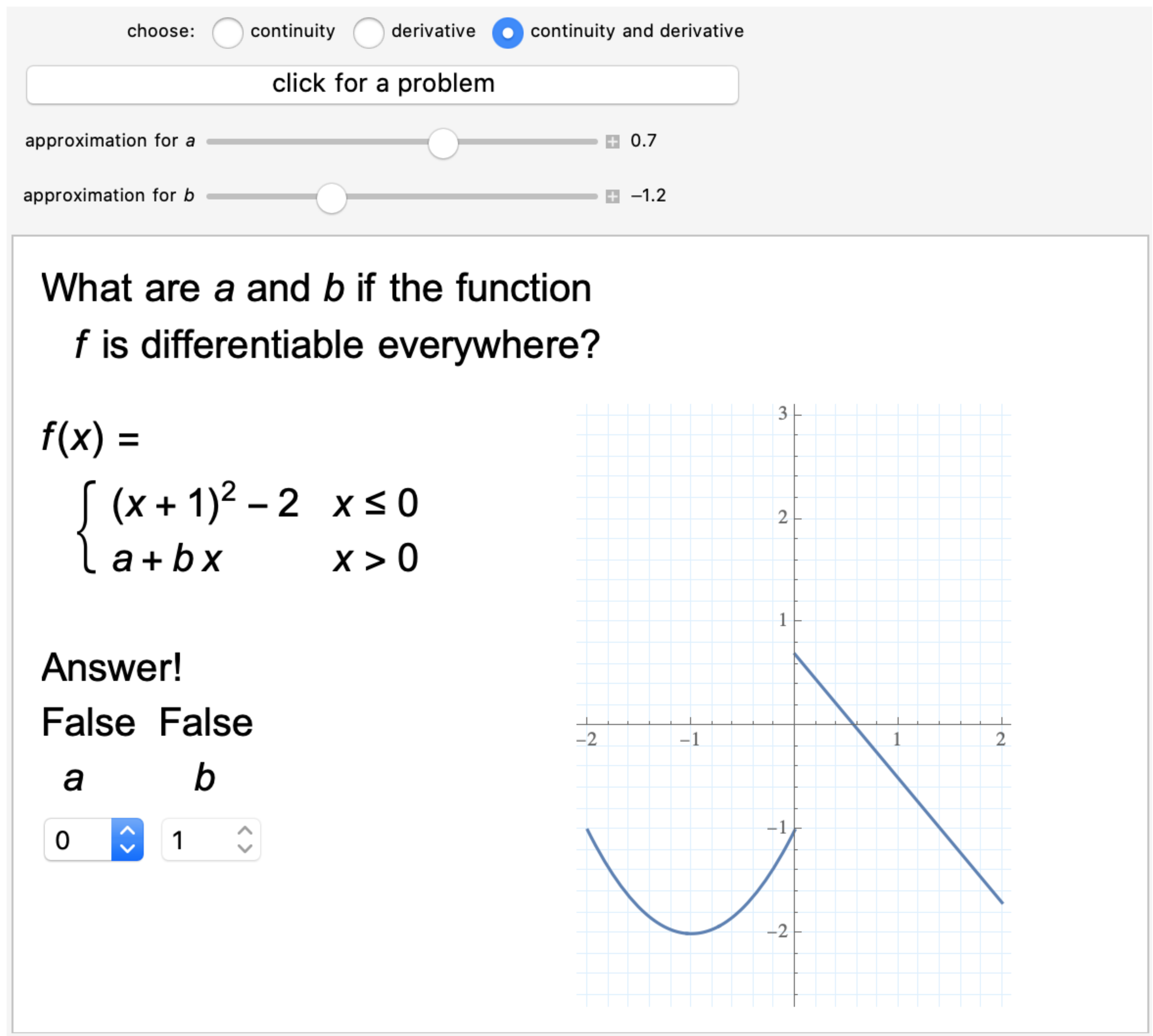
The differentiation microscope for $f(x) = |x^2 + x|$

Applets

Other authors' applets

§3.1

- When is a piecewise function differentiable? (*Author*: Izidor Hafner)

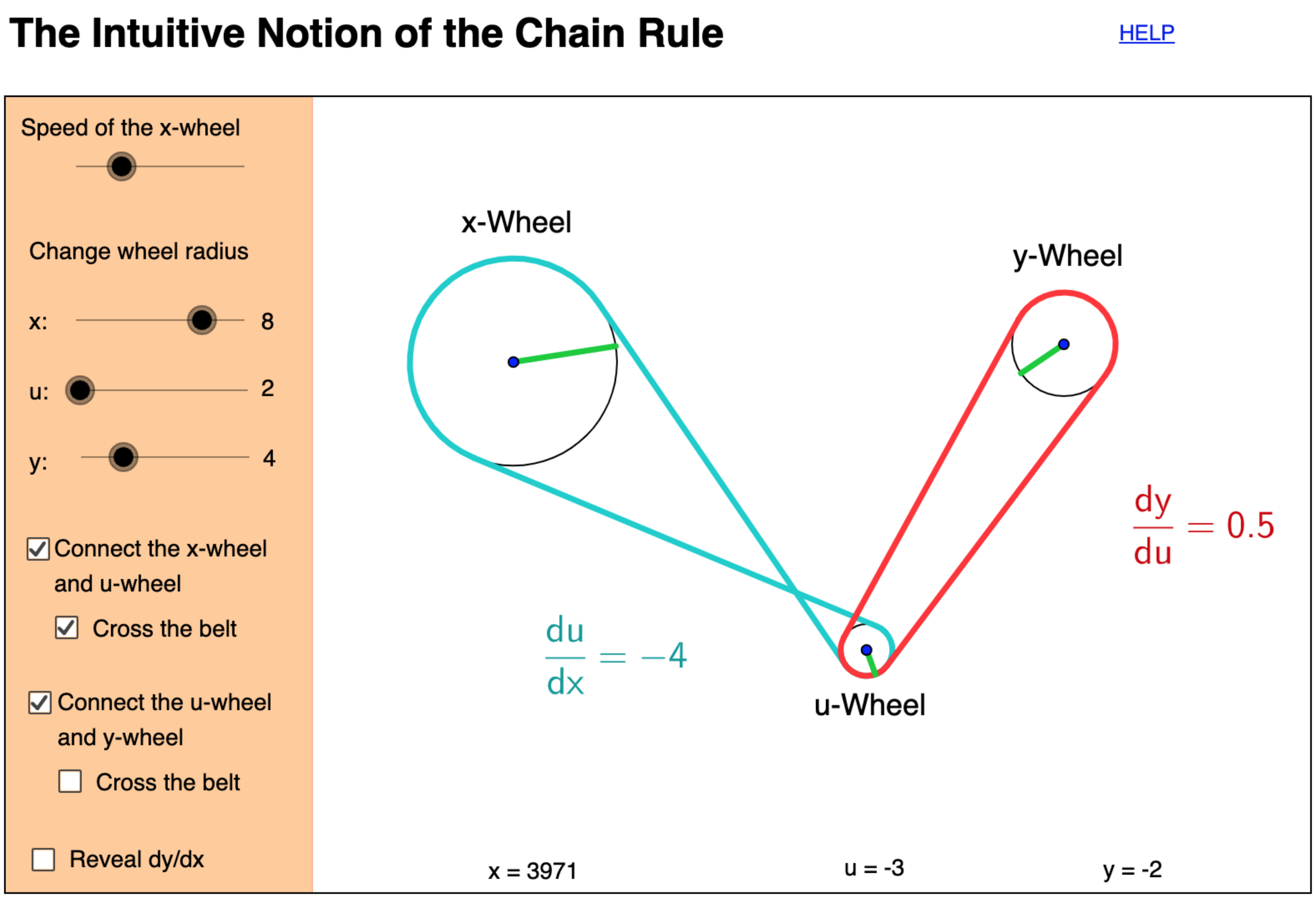


Applets

Other authors' applets

§3.4

- Wheels and belts (*Author*: Marc Renault, Shippensburg Unversity)

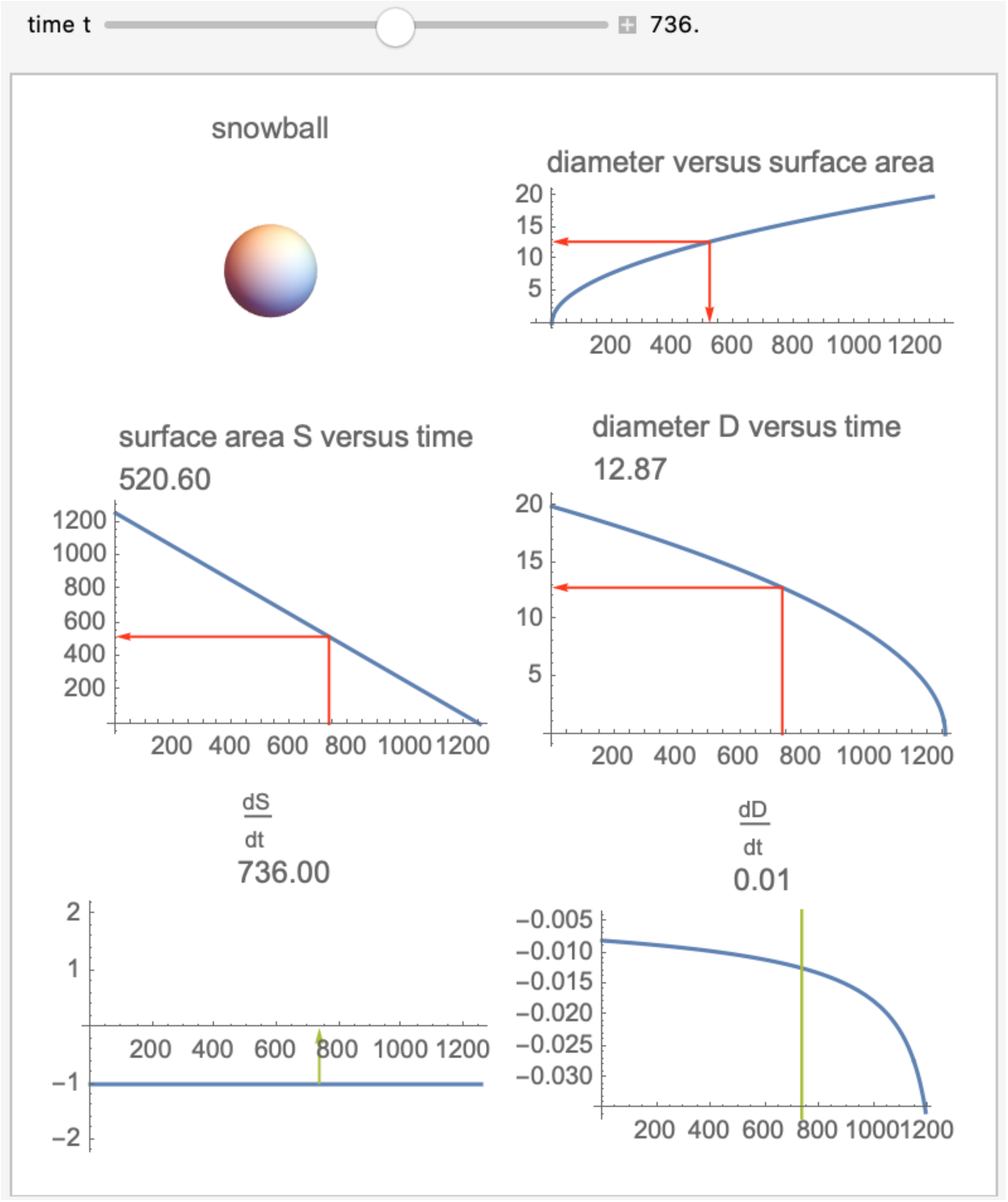


Applets

Other authors' applets

§3.4

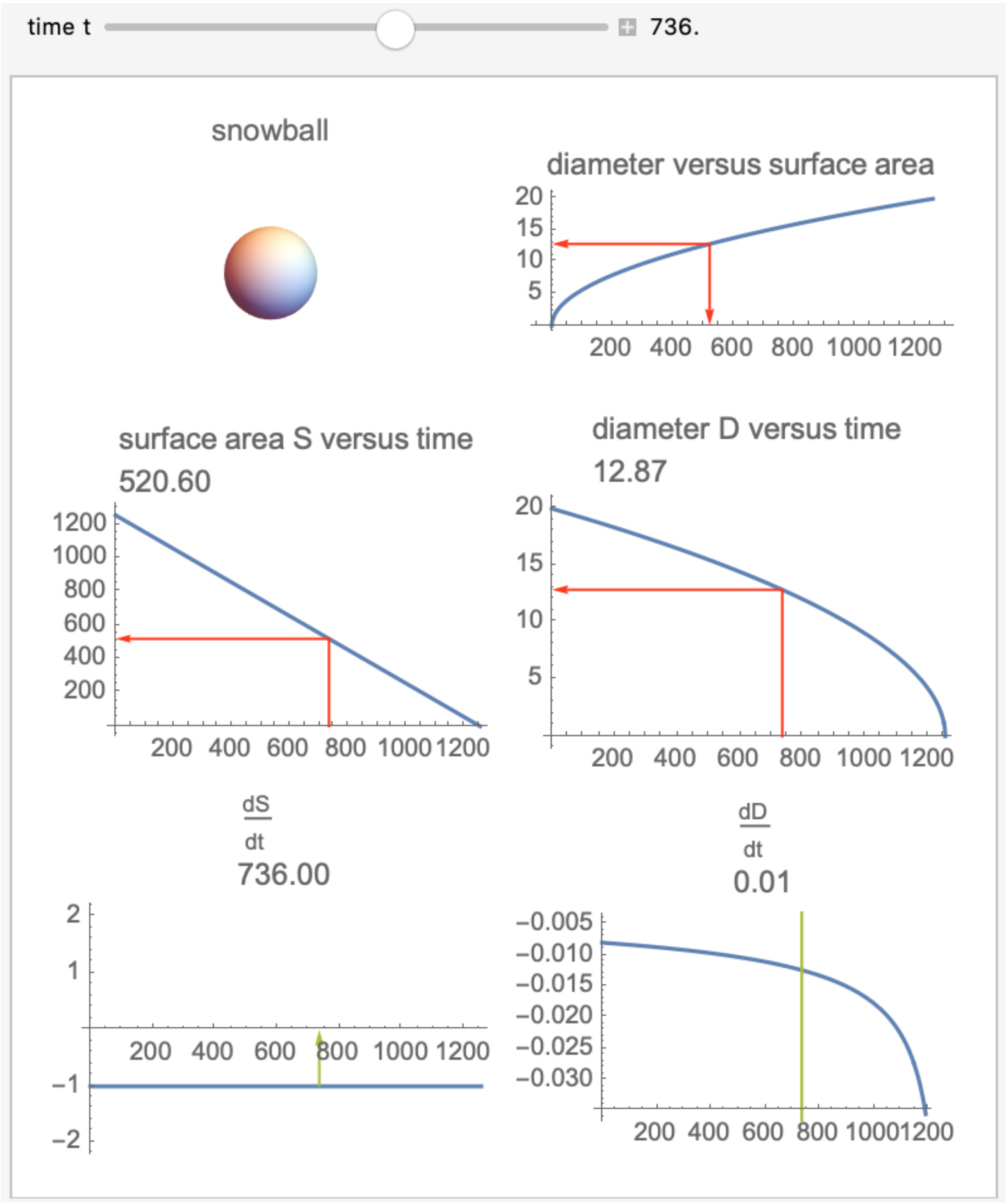
- A snowball's rate of change (*Authors*: Cindy Piao and Karen Ye, Torrey Pines High School)



Other authors' applets

§3.4

- A snowball's rate of change (*Authors*: Cindy Piao and Karen Ye, Torrey Pines High School)

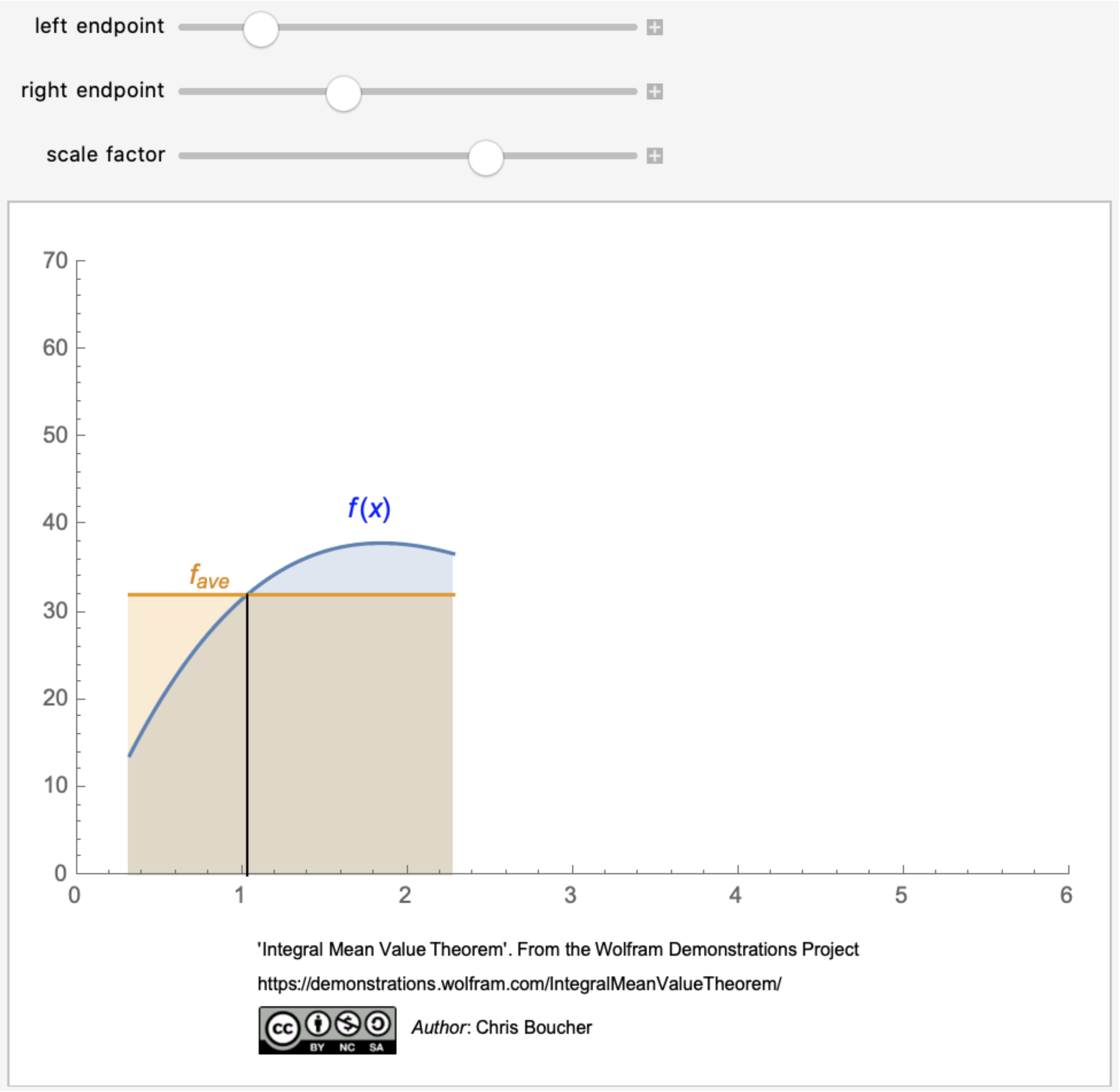


Applets

Other authors' applets

§5.2

- Mean Value Theorem for Integrals (*Author*: Chris Boucher)



Distribution

Wolfram Demonstrations Project

Pros:

- Share with other educators and users of Mathematica
- Provides “permanent” links to your uploaded submissions

Cons:

- Privately curated archive, may reject submissions
- Educator has limited control over how content is presented on Wolfram’s website
- Won’t host non-Mathematica materials

Distribution

OpenALG/Manifold

Pros:

- Share across USG and beyond
- USG and educator have control over how content is presented
- Can host a variety of file types bundled as a single project

Caveat:

- Not a replacement for a course management system like Brightspace/iCollege