The Markov-Dubins problem with free terminal direction in a nonpositively curved cube complex

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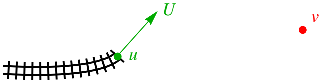
- (Markov, 1889): Formulated the problem with $X = \mathbb{R}^2$ in a little-known paper

Markov-Dubins problems

Markov-Dubins problem with free terminal direction

Find the shortest path between two points u, v in a space X, given a prescribed initial direction U and prescribed minimal turning radius $R > 0$.

- (Markov, 1889): Formulated the problem with $X = \mathbb{R}^2$ in a little-known paper
- Practical application: How can an existing length of railroad track be joined to a given destination, using as little new track as possible?



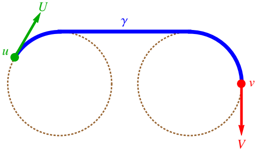
- Initial heading and position fixed; direction at the destination is not specified
- Minimal turning radius was needed to prevent derailment
- Problem seems to have been largely forgotten until the 1950s

Markov-Dubins problems

Markov-Dubins problem with free terminal direction

Find the shortest path between two points u, v in a space X, given a prescribed initial direction U and prescribed minimal turning radius $R > 0$.

- (Dubins, 1957): Solves the problem with prescribed initial and terminal direction, $X = \mathbb{R}^2$
- Finds that a shortest piecewise twice-differentiable solution always exists
- Length-minimizer is made up of at most three subarcs, each an arc of a circle of radius R or a line segment



Markov-Dubins problems

Markov-Dubins problem with free terminal direction

Find the shortest path between two points u, v in a space X, given a prescribed initial direction U and prescribed minimal turning radius $R > 0$.

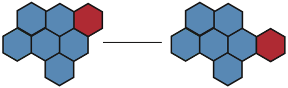
- (1960s–2000s): Other variations studied in robotics, game theory, differential geometry, avionics
  - Variations all take X to be a Riemannian manifold, usually of dimension 2 or 3
- For us, X will be a nonpositively curved cube complex
  - More practical than it may appear…

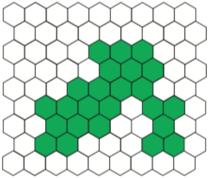
Markov-Dubins problems

Markov-Dubins problem with free terminal direction

Find the shortest path between two points u, v in a space X, given a prescribed initial direction U and prescribed minimal turning radius $R > 0$.

- For us, X will be a nonpositively curved cube complex
  - (Ghrist, 2002): Applied comparison geometry to reconfiguration problems for metamorphic robots (aggregates capable of changing shape through the independent motion of their constituent cells)





Two metamorphic systems composed of hexagonal cells.
A cell on the boundary of the aggregate may pivot if unobstructed (LEFT).

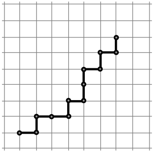
Figures from Ghrist and Peterson, “The geometry and topology of reconfiguration” (2007)

Markov-Dubins problems

Markov-Dubins problem with free terminal direction

Find the shortest path between two points u, v in a space X, given a prescribed initial direction U and prescribed minimal turning radius $R > 0$.

- For us, *X* will be a nonpositively curved cube complex
  - (Ghrist and Peterson, 2007): Uses theoretical framework of 2002 paper to describe a wide range of dynamical systems
    - Articulated robotic limb



The robotic arm of Ghrist and Peterson.

Figure from Ghrist and Peterson, "The geometry and topology of reconfiguration" (2007)

Markov-Dubins problems

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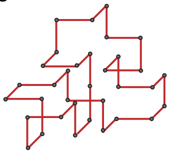

Model of a protein chain as a piecewise-linear chain in a cubical lattice.

Figure from Ghrist and Peterson, "The geometry and topology of reconfiguration" (2007)

Markov-Dubins problems

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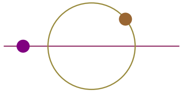
  - (Ghrist and Peterson, 2007): Uses theoretical framework of 2002 paper to describe a wide range of dynamical systems
    - Articulated robotic limb
    - Protein folding
    - Industrial track robots



Robots moving along tracks in a factory floor.

Example of a reconfigurable system

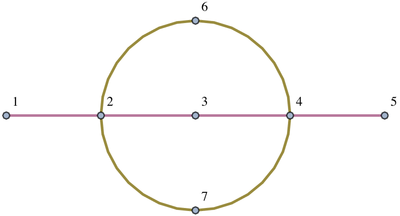
Reconfiguration problem

Move the robots from their given current positions to prescribed new positions in as short a time as possible while avoiding collisions.

Two robots moving along tracks on a factory floor

Graph-theoretic formulation of the reconfiguration problem

1. Discretize the two tracks, subdividing each into finitely many edges. The result is the **workspace graph** $\mathcal{W}$.



Vertices of the transition graph

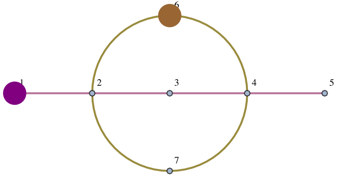
Graph-theoretic formulation of the reconfiguration problem

1. Discretize the two tracks, subdividing each into finitely many edges. The result is the **workspace graph** $\mathcal{W}$.
2. Construct the **transition graph** $\mathcal{T}$.
   - Records allowable configurations/states and allowable transitions between states

Vertices of the transition graph

Graph-theoretic formulation of the reconfiguration problem

1. Discretize the two tracks, subdividing each into finitely many edges. The result is the **workspace graph** $\mathcal{W}$.
2. Construct the **transition graph** $\mathcal{T}$.
   - The vertices of $\mathcal{T}$ are states of the system, represented as labelings of Vert$(\mathcal{W})$.
   - In our example, each vertex of $\mathcal{T}$ is a function $u : \{1, 2, 3, 4, 5, 6, 7\} \to \{\circ, \bullet, \bullet\}$.



A vertex of $\mathcal{T}$

Generators of a reconfigurable system

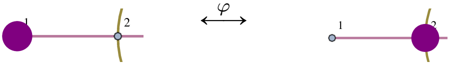
- Edges of $\mathcal{T}$
  - A set $\mathcal{G}$ of pairs of inverse elementary moves is specified.

    Such a pair is called a **generator**.
  - In our example, each elementary move slides one robot on its track to an unoccupied adjacent vertex.

    A generator φ is defined by the following pair of moves:

    If vertex 1 is occupied by ● and vertex 2 is unoccupied, move ● to vertex 2.

    If vertex 2 is occupied by ● and vertex 1 is unoccupied, move ● to vertex 1.

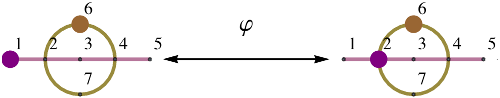


Edges of the transition graph

- Edges of $\mathcal{T}$
  - Each generator φ is represented as a pair of labelings of a subset $\mathcal{S}$ of Vert$(\mathcal{W})$. To **apply** φ to a state u means to redefine u on $\mathcal{S}$, obtaining a new labeling

$$\varphi[u] : \{1, 2, 3, 4, 5, 6, 7\} \to \{\circ, \bullet, \bullet\}.$$

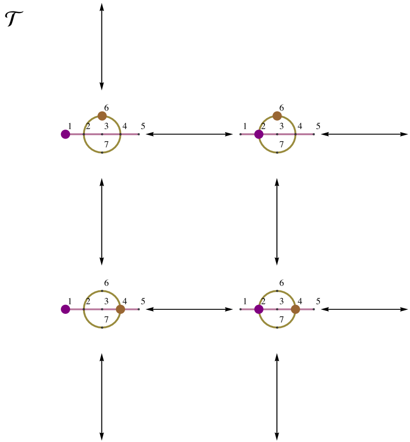
  - Two states $u, v \in$ Vert$(\mathcal{T})$ are joined by an edge in $\mathcal{T}$ if some generator $\varphi \in \mathcal{G}$ toggles the system between states u and v.



  - A collection of states that is closed under the application of all generators is an **(abstract) reconfigurable system**.

Transition graph

The transition graph $\mathcal{T}$ is analogous to the Cayley graph of a group, but need not be homogeneous: some generators may not be applicable to some states.



Graph-theoretic formulation of the reconfiguration problem

Graph-theoretic formulation of the reconfiguration problem

1. Discretize the two tracks, subdividing each into finitely many edges. The result is the **workspace graph** $\mathcal{W}$.
2. Construct the transition graph $\mathcal{T}$.
3. Find the shortest path from an initial state $u \in \text{Vert}(\mathcal{T})$ to the goal state $v \in \text{Vert}(\mathcal{T})$.

   Such a path corresponds to a sequence of elementary moves that reconfigures the system from state u to state v.

Drawback of graph-theoretical formulation

- A shortest path in $\mathcal{T}$ need not be an efficient reconfiguration strategy
- Transition graph does not encode information about which moves can be applied concurrently

Graph-theoretic formulation of the reconfiguration problem

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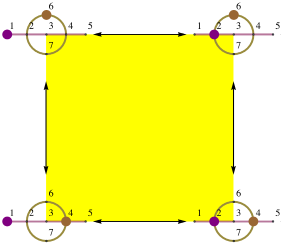
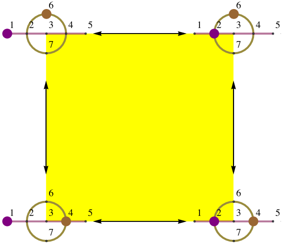
Geometric formulation of the reconfiguration problem



Use cubes to encode concurrency

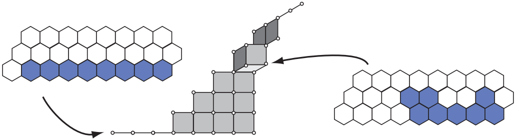
- We say that k generators **commute** at a state u if they can be applied to u simultaneously, and if the resulting configuration is independent of the order in which they are applied.
- Wherever the 1-skeleton $Q^{(1)}$ of a k-cube appears in $\mathcal{T}$, attach a k-cube if for each vertex u of $Q^{(1)}$, the generators corresponding to the edges incident with u commute at u.

Geometric formulation of the reconfiguration problem



By attaching cubes to the transition graph as described, the configuration space of a reconfigurable system is realized as a cube complex called the **state complex.**

The state complex of a reconfigurable system

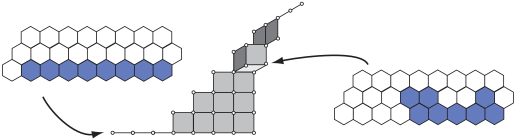


State complex for a metamorphic robotic system

composed of pivoting hexagonal tiles (Ghrist-Peterson, 2007)

- Interior points of a cube are intermediate stages of a transition between states.
- A path along a k-cube's diagonal represents the simultaneous application of the k commuting generators corresponding to the k parallelism classes of the cube's edges.
- A path from u to v in the state complex determines a strategy for reconfiguring the system from state u to state v.

The state complex of a reconfigurable system



State complex for a metamorphic robotic system composed of pivoting hexagonal tiles (Ghrist-Peterson, 2007)

(Ghrist, 2002): The state complex of a reconfigurable system is a nonpositively curved cube complex.

When *u* and *v* are fixed, efficient algorithms exist for finding the shortest path between them.

- (Ardila-Owen-Sullivant, 2011): General nonpositively curved cube complex
- (Chepoi-Maftuleac, 2012): Nonpositively curved rectangular complexes

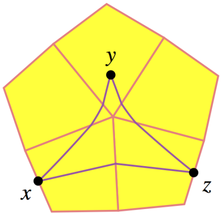
Fault tolerance

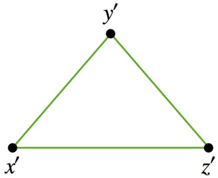
But in a real world environment, changing circumstances in the physical workspace may intervene to make a reconfiguration strategy that is already in progress impossible to complete.

- Suppose a goal state has been prescribed, but an obstruction prevents us from attaining it. A new goal state in the state complex may then be prescribed.
- It is inefficient to bring the system to a halt whenever a new strategy is prescribed, and impractical to instantaneously follow the new strategy without stopping.
- We therefore seek a solution to the problem of finding a shortest path in the state complex with a given initial direction.
- In order to limit the stress placed on the system's physical components, we impose a bound on the path's curvature.

Before we give a formal statement of our central problem, we will briefly review the definition of a *nonpositively curved geodesic space*. . .

Nonpositive curvature
Comparison triangles





(LEFT:) A geodesic triangle Δxyz in a square complex, and (RIGHT:) a comparison triangle in $\mathbb{R}^2$ for Δxyz

A metric space (X, d) is a **geodesic space** if every $x, y \in X$ can be joined by a path in X of length $\ell = d(x, y)$, called a **geodesic**.

- A geodesic from x to y will be denoted by $[xy]$.

Let x, y, z be three distinct points in a geodesic space. Then

$$\Delta xyz := [xy] \cup [yz] \cup [zx]$$

is a **geodesic triangle**, and a **comparison triangle** $\Delta x'y'z'$ for Δxyz is a triangle in the Euclidean plane with corresponding sides equal in length.

Nonpositive curvature
Thin triangles

A geodesic triangle Δxyz is **thin** if the distance between any two points on Δxyz is no larger than the distance between the corresponding points on a comparison triangle:

$$d(p,q) \leqslant d(p',q').$$

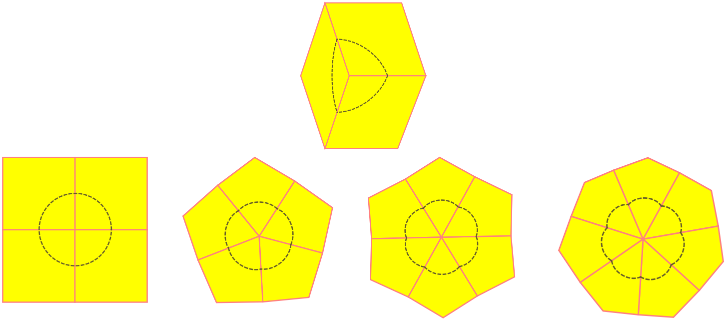
A geodesic space X is

- **nonpositively curved (NPC)** at a point w if all geodesic triangles sufficiently near w are thin,
- **nonpositively curved** if X is nonpositively curved at every point,
- **CAT(0)** if X is simply connected and nonpositively curved.

Nonpositive curvature

Nonpositively curved square complexes

The square complex obtained by arranging *d* copies of the unit square $[0,1] \times [0,1]$ cyclically around a central vertex is nonpositively curved if $d \geqslant 4$.



A positively curved (*top row*) and some nonpositively curved (*bottom row*) piecewise Euclidean square complexes

Boundaries of metric balls (*dashed*) about center vertex are unions of arcs of Euclidean circles

Central problem: Markov-Dubins problem with free terminal direction in a NPC cube complex

Markov-Dubins problem with free terminal direction

Let $\kappa > 0$. Given initial and terminal positions u and v in a nonpositively curved cube complex X, find the shortest unit-speed path γ in X from u to v such that

- γ *has prescribed initial direction* $U \in \text{link}(u)$,
- $|\gamma''| \leqslant \kappa$ *a.e. in local coordinates, and*
- γ *is smooth (has turning angle 0) at breakpoints.*

Sufficient collection of paths

Existence theorem

- Preliminary:

  Define a collection $\mathcal{C}$ of "admissible" unit-speed paths γ so that admissible paths satisfy the boundary conditions (u, U, v) and the curvature constraint with $\kappa > 0$.

- Then show that $\mathcal{C} \neq \varnothing \implies \mathcal{C}$ contains a shortest path.

- How should we define "admissible"?
  - Minimally, want $\mathcal{C}^1$ in local coordinates
  - Twice-differentiable in local coordinates, with curvature $|\gamma''| \leqslant \kappa$?

Sufficient collection of paths

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  - (Dubins, 1957): For certain choices of $\kappa > 0$, $U \in S^1$, and $u, v \in \mathbb{R}^2$, the collection $\mathcal{D}$ of twice-differentiable unit-speed paths $\gamma : [a, b_\gamma] \to \mathbb{R}^2$ with

    $$\gamma(a) = u, \quad \gamma'(a) = U, \quad \gamma(b_\gamma) = v, \quad |\gamma''| \leqslant \kappa$$

    does not contain an element of minimum length...

Sufficient collection of paths

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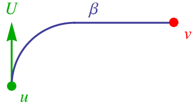
does not contain an element of minimum length.

But for any choice of $\kappa > 0$, u, U, and v, there exists a $\mathcal{C}^1$ and piecewise twice-differentiable path β with length $\ell(\beta) = \inf_{c \in \mathcal{D}} \ell(c)$.

Example:



Pick

$$\kappa = 1, \quad u = (0,0), \quad U = (0,1), \quad v = (3,1),$$

and let β be the shortest CL path in $\mathbb{R}^2$ satisfying these boundary conditions and curvature bound.

- A **CL path** is the $\mathcal{C}^1$ concatenation of a circular arc and a line segment.

Sufficient collection of paths

- (Dubins, 1957): For certain choices of $\kappa > 0$, $U \in S^1$, and $u, v \in \mathbb{R}^2$, the collection $\mathcal{D}$ of twice-differentiable unit-speed paths $\gamma : [a, b_\gamma] \to \mathbb{R}^2$ with

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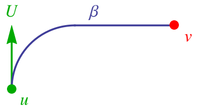


Then

- Every $\gamma \in \mathcal{D}$ has $\ell(\gamma) > \ell(\beta)$.
- For any $\varepsilon > 0$, there exists $\gamma \in \mathcal{D}$ with $\ell(\beta) < \ell(\gamma) < \ell(\beta) + \varepsilon$.

Sufficient collection of paths

How should we define "admissible"?

- $\mathcal{C}^1$ in local coordinates, with κ-Lipschitz derivative
  - Rules out abrupt changes in direction which would put stress on moving parts of the system
  - Permits CL paths
  - Can use Dubins' characterization of optimal paths to describe optimal paths contained in a cell

We now define *piecewise-Lipschitz differentiability* for curves in a cube complex.

Piecewise Lipschitz-differentiable curves in a cube complex

Let $\gamma : [a, b] \to X$ be a path in a cube complex X, and let

$$a = t_1 < t_2 < \cdots < t_m = b$$

be a partition of $[a, b]$. A sequence $(Q_k)_{k=1}^{m-1}$ of cells in X is a **cube path for γ with breakpoints** $(t_k)_{k=1}^{m}$ if

$$\gamma([t_k, t_{k+1}]) \subset Q_k, \qquad Q_k \not\subset Q_{k+1}, \qquad Q_k \not\supset Q_{k+1}.$$

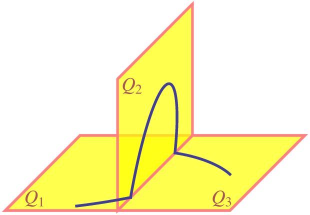
A cube path for a curve $\gamma : [a, \infty) \to X$ is defined similarly, taking $m = \infty$.

We call each

$$\gamma_k := \gamma\big|_{[t_k, t_{k+1}]}$$

a **segment** of γ.

- Note $\gamma(t_k) \in Q_{k-1} \cap Q_k$
- **edgewise** cube path: each $Q_k \cap Q_{k+1}$ is an edge
- **locally monotone square path**: $Q_{k-1} \cap Q_k \neq Q_k \cap Q_{k+1}$ for each suitable k, and $\dim Q_k = 2$ for all k



A square path which is not locally monotone

Piecewise Lipschitz-differentiable curves in a cube complex

A path in $\mathbb{R}^N$ is $\boldsymbol{\kappa}$-$\boldsymbol{\mathcal{C}^{1,1}}$ (or **κ-Lipschitz differentiable**) if its derivative exists and is κ-Lipschitz.

Let X be a cube complex, let $\kappa > 0$, and let $M \in \mathbb{N}$ ($M \geqslant 2$). A path

$$\gamma : [a, b] \to X$$

is $\boldsymbol{\kappa}$-$\boldsymbol{\mathcal{C}^{1,1}(M)}$ (or **κ-Lipschitz differentiable with at most M breakpoints**) if there exists a cube path $(Q_k)_{k=1}^{m-1}$ for γ with breakpoints

$$a = t_1 < t_2 < \cdots < t_m = b \quad (m \leqslant M)$$

such that each segment

$$\gamma_k : [t_k, t_{k+1}] \to Q_k$$

is κ-$\mathcal{C}^{1,1}$ in coordinates.

Length-minimal element of $\mathcal{C}(\kappa, M, u, v, U)$

Let X be a nonpositively curved locally finite cube complex. Fix

- u, v in X such that $u \neq v$,
- a unit tangent vector U to $p(C_\lambda)$ at $p_\lambda(u)$ for some λ,
- $\kappa > 0$, and
- $M \in \mathbb{N}$ ($n \geqslant 2$).

Let

$$\mathcal{C}(\kappa, M, u, v, U)$$

be the set of κ-$\mathcal{C}^{1,1}(M)$ unit-speed paths $\gamma : [a, b_\gamma] \to X$ with

$$\gamma(a) = u, \quad \gamma(b_\gamma) = v, \quad \gamma'(a) = U.$$

Theorem

If $\mathcal{C} = \mathcal{C}(\kappa, M, u, v, U)$ is nonempty, then $\mathcal{C}$ contains a path β of minimal length among all paths in $\mathcal{C}$.

Smoothness at breakpoints

The proof of the theorem uses only advanced calculus and Arzela-Ascoli. We begin with a sequence of admissible paths $\gamma_n \in \mathcal{C}$ such that

$$\ell(\gamma_n) \searrow \inf_{c \in \mathcal{C}} \ell(c),$$

and repeatedly pass to subsequences so that

- there is a single cube path $(Q_k)_{k=1}^{\infty}$ with breakpoints $(t_k)_{k=1}^{\infty}$ for all γ_n, and
- for each k, each of $\left(\gamma_n\big|_{[t_k, t_{k+1}]}\right)_{n=1}^{\infty}$, $\left(\gamma_n'\big|_{[t_k, t_{k+1}]}\right)_{n=1}^{\infty}$ converge.

We then show that the uniform limit γ of the γ_n is in $\mathcal{C}$.

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and repeatedly pass to subsequences so that

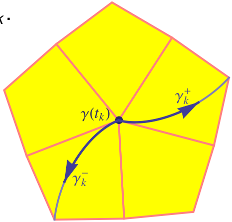
- there is a single cube path $(Q_k)_{k=1}^{\infty}$ with breakpoints $(t_k)_{k=1}^{\infty}$ for all γ_n, and
- for each k, each of $\left(\gamma_n|_{[t_k,t_{k+1}]}\right)_{n=1}^{\infty}$, $\left(\gamma_n'|_{[t_k,t_{k+1}]}\right)_{n=1}^{\infty}$ converge.

We then show that the uniform limit γ of the γ_n is in $\mathcal{C}$.

More is needed to prove the existence of a solution to the Markov-Dubins problem: we want our paths to have zero **turning angle**

$$\pi - \angle(\gamma_k^-, \gamma_k^+) \in [0, \pi], \qquad \gamma_k^- := \overline{\gamma|_{(t_k-\varepsilon, t_k]}}, \qquad \gamma_k^+ := \gamma|_{[t_k, t_k+\varepsilon)},$$

for each interior breakpoint t_k.



Smoothness at breakpoints

The proof of the theorem uses only advanced calculus and Arzela-Ascoli. We begin with a sequence of admissible paths $\gamma_n \in \mathcal{C}$ such that

$$\ell(\gamma_n) \searrow \inf_{c \in \mathcal{C}} \ell(c),$$

and repeatedly pass to subsequences so that

- there is a single cube path $(Q_k)_{k=1}^{\infty}$ with breakpoints $(t_k)_{k=1}^{\infty}$ for all γ_n, and
- for each k, each of $\left(\gamma_n\big|_{[t_k, t_{k+1}]}\right)_{n=1}^{\infty}, \left(\gamma_n'\big|_{[t_k, t_{k+1}]}\right)_{n=1}^{\infty}$ converge.

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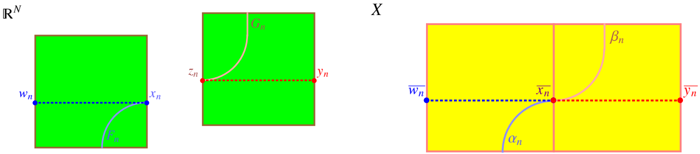
for each interior breakpoint t_k.

- The property of having zero turning angle at breakpoints is preserved when passing to the limit.
- We carry out the same argument as for the previous theorem, but this time, we'll require that admissible paths have zero turning angle at interior breakpoints.

Existence result for Markov-Dubins problem with free terminal direction



Some technical issues and their solutions:

$\lim_{n\to\infty} \ell(\gamma_k^n) \neq 0$ $\lim_{n\to\infty} \ell(\gamma_{k+1}^n) \neq 0$	Construct geodesics with same directions as subarcs (*above fig.*) and use u.s.c. of $\angle$
$\lim_{n\to\infty} \ell\left(\gamma_1^n * \cdots * \gamma_k^n\right) = 0$	Reparametrize and delete $(Q_i)_{i=1}^k$ from cube path
$\lim_{n\to\infty} \ell(\gamma_k^n) \neq 0$ $\lim_{n\to\infty} \ell(\gamma_{k+1}^n) = 0$ $\lim_{n\to\infty} \ell(\gamma_{k+2}^n) \neq 0$	Total curvature is lower semicontinuous, $\therefore \tau_k + \tau_{k+2} = \lim_n \tau^n[t_k, t_{k+3}] \geqslant \tau[t_k, t_{k+3}]$ $= \tau_k + \angle(\bar{\gamma}_k, \gamma_{k+2}) + \tau_{k+2}, \quad \therefore \angle(\bar{\gamma}_k, \gamma_{k+2}) = 0.$
$\lim_{n\to\infty} \ell(\gamma_k^n * \cdots * \gamma_m^n) = 0$	Reparametrize and delete $(Q_i)_{i=k}^{m-1}$ from cube path

Here $\gamma_k^n = \gamma_n\big|_{[t_k, t_{k+1}]}$.

Existence result for Markov-Dubins problem with free terminal direction

We say a κ-$\mathcal{C}^{1,1}(M)$ path γ is **smooth at breakpoints** if there exists a cube path $(Q_k)_{k=1}^{m-1}$ ($m \leqslant M$) for γ with breakpoints $(t_k)_{k=1}^{m}$ such that γ has zero turning angle at $\gamma(t_k)$ for $1 < k < m$.

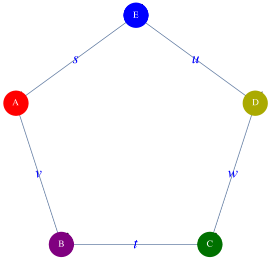
For κ, M, u, v, U as above, write

$$\begin{aligned} \mathcal{C}_0 &= \mathcal{C}_0(\kappa, M, u, v, U) \\ &= \{\gamma \in \mathcal{C}(\kappa, M, u, v, U) : \gamma \text{ is smooth at breakpoints}\}. \end{aligned}$$

Theorem

If $\mathcal{C}_0$ is nonempty, then $\mathcal{C}_0$ contains a path β of minimal length among all paths in $\mathcal{C}_0$.

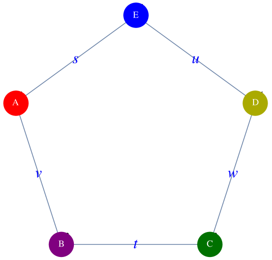
Numerical solution of the Markov-Dubins problem with free terminal vector



We will now outline an algorithm for numerically finding the shortest CL path between two points with prescribed initial direction in a NPC square complex.

- Markov found that the solution to the Markov-Dubins problem with free terminal direction in $\mathbb{R}^2$ always exists and is a CL path.
- In a NPC square complex, an optimal CL path with prescribed boundary conditions u, U, v and curvature bound $\kappa > 0$ need not exist, but if it does, our algorithm will find it.

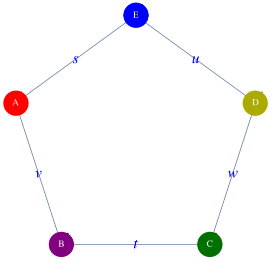
Numerical solution of the Markov-Dubins problem with free terminal vector



The particular square complex we'll use to visualize the algorithm arises from the reconfigurable system defined as follows.

- Five distinctly labeled checkers are placed at the vertices of a pentagon.
- The generators are transpositions of the checkers on an edge.
- Generators commute iff the corresponding edges are disjoint, and no set of 3 edges is disjoint.

Numerical solution of the Markov-Dubins problem with free terminal vector



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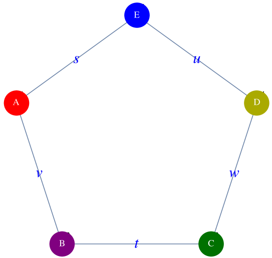
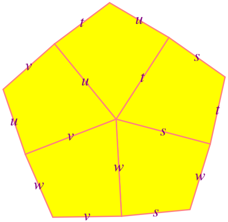




- Thus each vertex in the state complex is incident with exactly five squares arranged cyclically.
- The transition graph is the Cayley graph of the right-angled Coxeter system

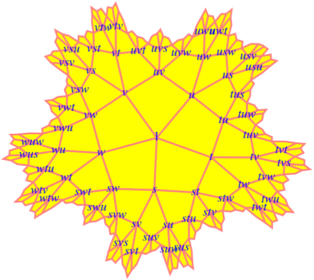
$$S = \{s, t, u, v, w\},$$

$$W = \langle S \mid s^2 = t^2 = u^2 = v^2 = w^2 = (st)^2 = (tu)^2 = (uv)^2 = (vw)^2 = (ws)^2 = 1 \rangle.$$

Numerical solution of the Markov-Dubins problem with free terminal vector

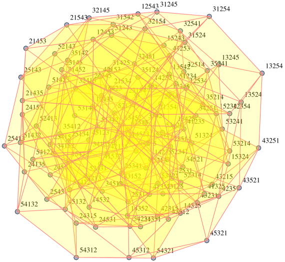
- The state complex is a closed orientable surface of genus 16, whose universal cover is the Davis complex of (W, S).

$\mathcal{X}_5$



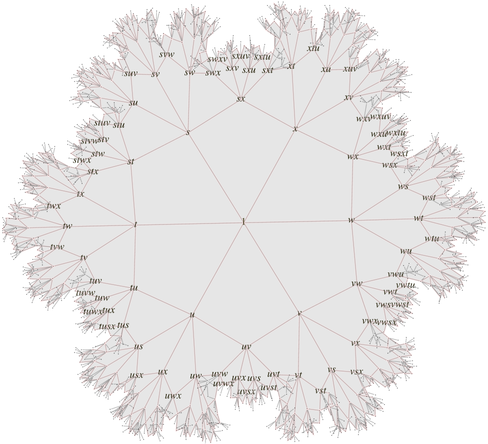
$\longrightarrow$

state complex



- The universal cover of the state complex is a space we call the **5-plane $\mathcal{X}_5$**.

Definition of the d-plane $\mathcal{X}_d$



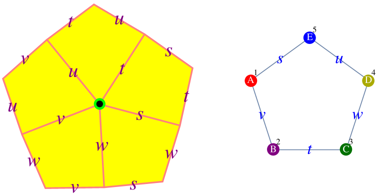
Definition. The ***d*-plane** $\mathcal{X}_d$ ($d \geqslant 4$) is a simply connected surface without boundary that is a piecewise Euclidean square complex with a d-regular graph as its 1-skeleton.

- $\mathbb{E}^2 := \mathcal{X}_4$.

Fault handling is a Markov-Dubins problem

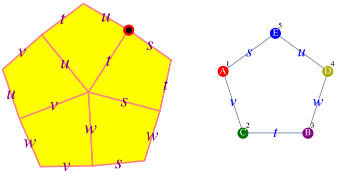
For an illustration of the problem, suppose the system begins in a given state,

Initial state



and a goal state is given. . .

Goal state
(*Swap B and C*)



Fault handling is a Markov-Dubins problem

. . . but a new goal state is prescribed before the original goal state has been attained.

At the instant when the new goal state is prescribed, we have a Markov-Dubins problem with prescribed initial position, initial direction, and terminal position.

To find CL paths in a NPC square complex, we will need an algorithm for finding geodesic paths.

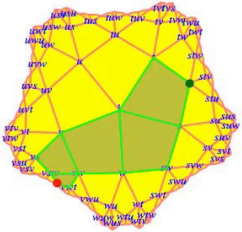
Chepoi-Maftuleac unfolding

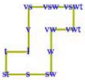
Chepoi and Maftuleac's algorithm for finding geodesics depends on the following result.

Theorem (Chepoi-Maftuleac, 2012)

The geodesic between two points u, v of a CAT(0) square complex X lies in a subcomplex K of X which is isometric to a monotone planar polygon. Moreover, K depends only on the choice of 2-cells containing u and v.

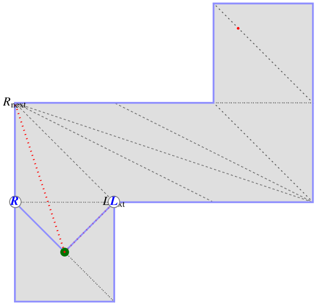
- This result reduces the problem of finding shortest paths in X to that of finding shortest paths in planar polygons: no shortcut from u to v is possible by leaving K.





Here, a polygon P is **monotone** if every axis-aligned line segment with endpoints in P is contained in P.

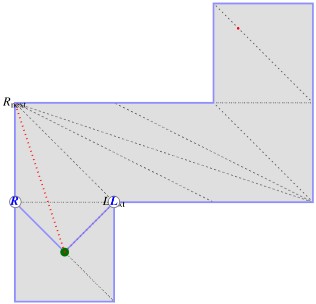
Lee-Preparata funnel algorithm



Many algorithms for finding shortest paths in a monotone polygon exist.
We use the classic “funnel algorithm” of Lee and Preparata.

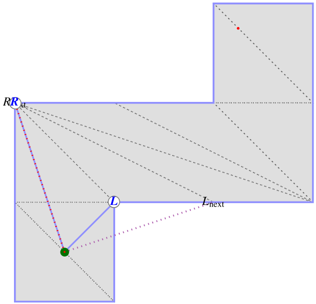
- A monotone polygon P can be triangulated by edges E_i with endpoints on the boundary of P
- The dual graph of the triangulation is a path
- The path determines an ordering of edges E_i that starts with the base of a triangle containing the initial point u (*green dot*) and ends with the base of a triangle containing the terminal point v (*red dot*)

Lee-Preparata funnel algorithm



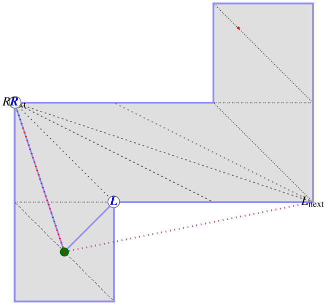
- The algorithm begins with a "funnel" F with legs $[uL]$ and $[uR]$ (*blue*), whose apex is the initial point u, where $[LR]$ is an edge of a triangle containing u
- The funnel F consists of the cone of lines of sight between $[uL]$ and $[uR]$ from u to the boundary of P
- If the terminal point v lies in the funnel F, we are done: $[uv]$ lies in P
- If not, we look at the next edge E'. If the resulting funnel is contained in the current funnel F, we accept E' as the base of the funnel F' for the next step

Lee-Preparata funnel algorithm



- If the next edge E' determines a funnel not contained in the current funnel, we reject it, set $F' = F$, and move to the next edge

Lee-Preparata funnel algorithm



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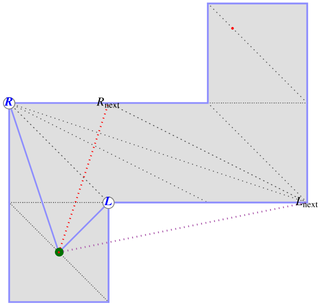

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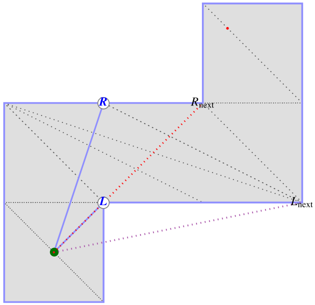
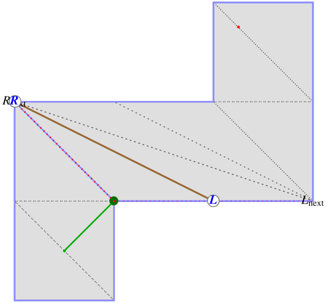


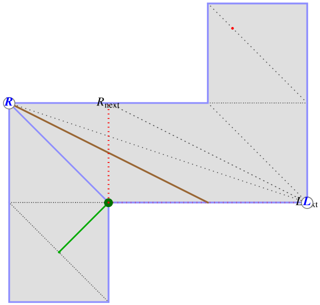
- If a leg of the funnel determined by the next edge crosses over or meets the opposite leg of the current funnel F, we have found a segment of the desired shortest path $[uv]$

Lee-Preparata funnel algorithm



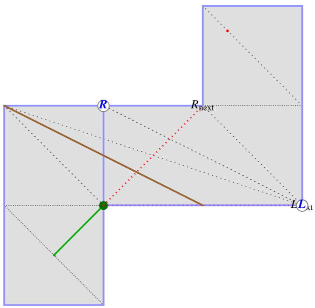
- We then reset u (*green dot*), choose a new initial edge E (*brown*) of the triangulation, and repeat the process, halting if v lies in the current funnel or if we have reached the final interior edge of the triangulation

Lee-Preparata funnel algorithm



- If the resulting funnel is contained in the current funnel F, we accept E' as the base of the funnel F' for the next step

Lee-Preparata funnel algorithm



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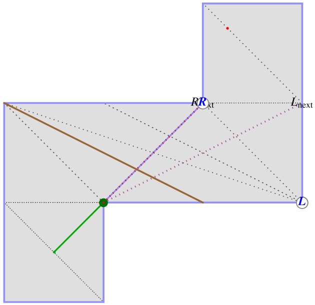

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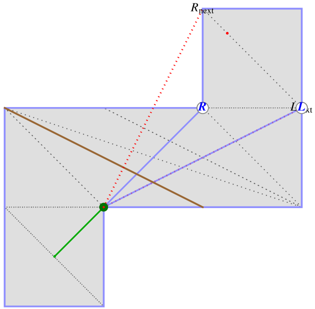
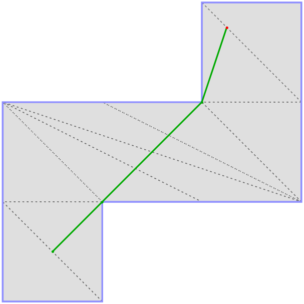
Lee-Preparata funnel algorithm

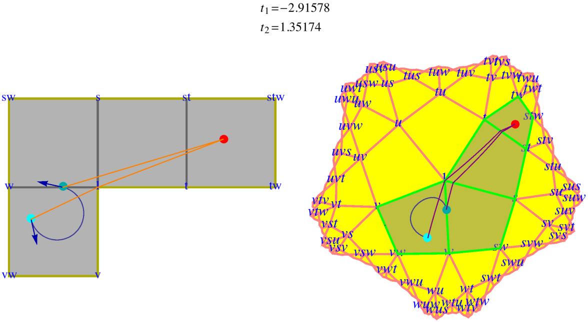


- Halt: we have reached the final interior edge of the triangulation
- The final segment begins with the current apex if v lies in the current funnel F, and begins with L or R (whichever is closer to v) if outside F

Lee-Preparata funnel algorithm



Determination of CL paths with the bisection method

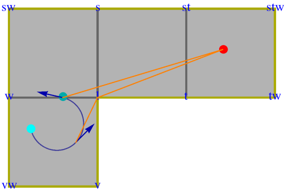


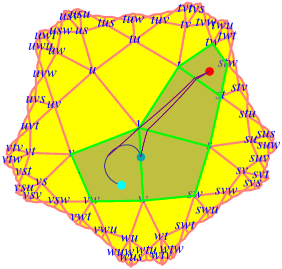
We determine the CL paths from u to v with initial direction U by the bisection method.

Determination of CL paths with the bisection method

$t_1 = -0.782022$

$t_2 = 1.35174$



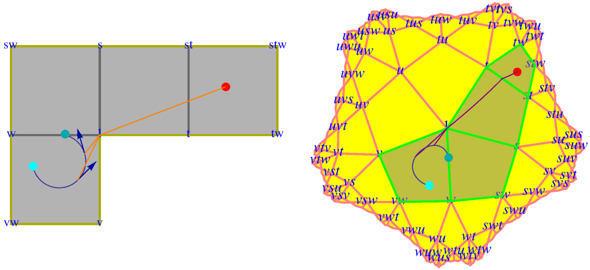


The directed angle Θ between a tangent to the initial arc, and the geodesic from the foot of the tangent to v, is a continuous function. . .

Determination of CL paths with the bisection method

$t_1 = -0.782022$

$t_2 = 0.284858$



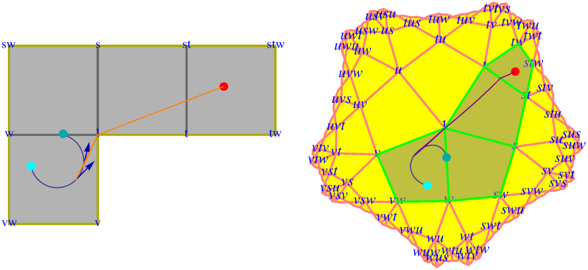
The directed angle Θ between a tangent to the initial arc, and the geodesic from the foot of the tangent to v, is a continuous function.

The zeroes of Θ correspond to CL paths from u to v.

Determination of CL paths with the bisection method

$t_1 = -0.782022$

$t_2 = -0.248582$

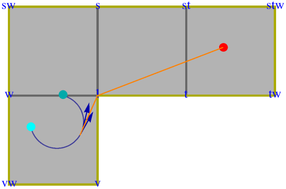


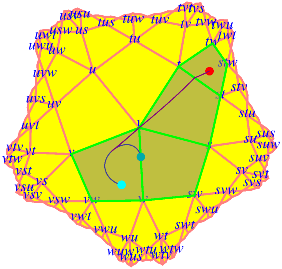
All CL paths from u to v with length less than some prescribed maximal length L can be found by finitely many applications of the bisection method.

Determination of CL paths with the bisection method

$t_1 = -0.515302$

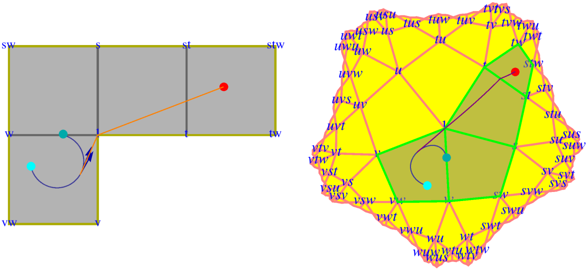
$t_2 = -0.248582$





Determination of CL paths with the bisection method

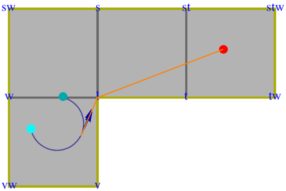
$t_1 = -0.515302$
$t_2 = -0.381942$

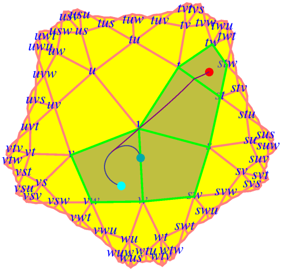


Determination of CL paths with the bisection method

$t_1 = -0.448622$

$t_2 = -0.381942$

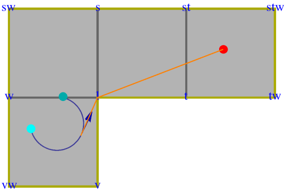


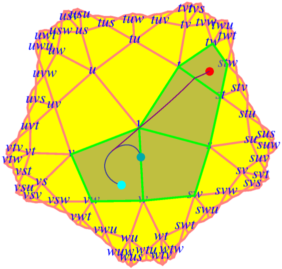


Determination of CL paths with the bisection method

$t_1 = -0.448622$

$t_2 = -0.415282$

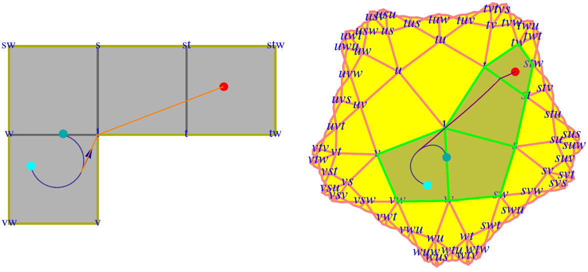




Determination of CL paths with the bisection method

$t_1 = -0.431952$

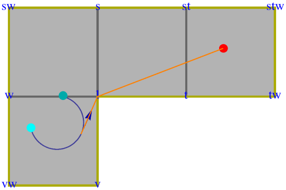
$t_2 = -0.415282$

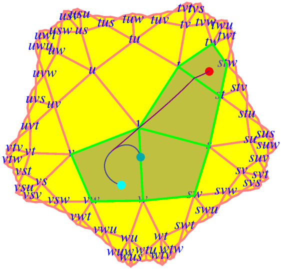


Determination of CL paths with the bisection method

$t_1 = -0.423617$

$t_2 = -0.415282$





Existence of CL paths

In the *d*-plane ($d \geqslant 5$), a CL path between two points with given initial direction and curvature constant $\kappa > 0$ need not exist.

To see why, we must look at the behavior of rays of constant curvature in $\mathcal{X}_d$

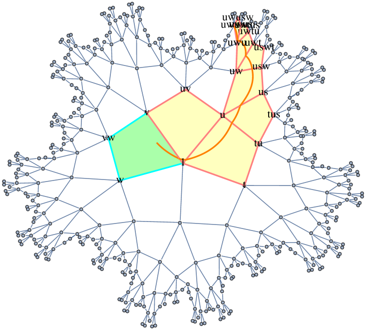
- By a **(topological) ray** in a space *X*, we mean a map of a half-line into *X*.
- Not necessarily embedded or geodesic

Existence of CL paths

In the d-plane ($d \geqslant 5$), a CL path between two points with given initial direction and curvature constant $\kappa > 0$ need not exist.

To see why, we must look at the behavior of rays of constant curvature in $\mathcal{X}_d$

- By a **(topological) ray** in a space X, we mean a map of a half-line into X.
- Not necessarily embedded or geodesic
- A ray in the d-plane which does not turn sharply enough cannot return to its starting point:

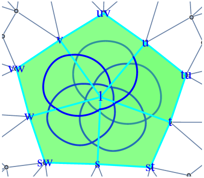


Classifying rays of constant curvature in $\mathcal{X}_d$

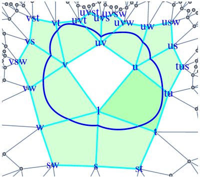
Numerical experiments reveal three possibilities for a ray

$$\gamma : [a, \infty) \to \mathcal{X}_d^* := \mathcal{X}_d \setminus \mathrm{Vert}(\mathcal{X}_d)$$

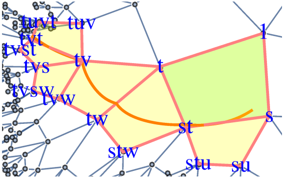
of constant curvature $\kappa > 0$...



Rose curves



Embedded circles



Proper rays

...whose behavior is not a function of κ when $d \geqslant 5$.

- A ray $[a, \infty) \to \mathcal{H}^2$ of constant curvature κ is proper if $\kappa \leqslant 1$, or a circle if $\kappa > 1$. Similarly for $\mathbb{R}^2$.
- $\exists\ \gamma_i : [a, \infty) \to \mathcal{X}_5$ of constant curvature $\kappa_i > 0$ ($i = 1, 2$) with γ_1 a circle and γ_2 a proper ray, while $0.895 = \kappa_1 < \kappa_2 = 0.968$.

Inaccessibility and properness

We can identify regions of the *d*-plane which are inaccessible by a CL path γ if the initial arc of γ is a subarc of a *proper* ray.

- A map is **proper** if the preimage of every compact set is compact.
- A ray $\gamma : [a, \infty) \to X$ **eventually never returns** to a subset S of X if for some $t > a$,
$$S \cap \gamma\big([t, \infty)\big) = \varnothing,$$
and $\gamma(s) \in S$ for some $s < t$.
- A ray is proper $\iff$ it eventually never returns to each cell it meets.

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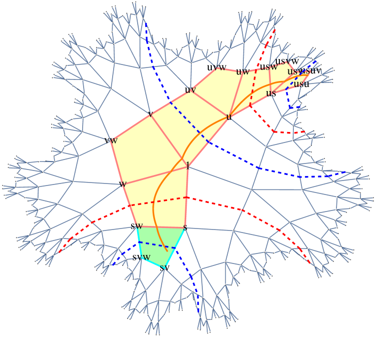

Rough idea: To show a ray in $\mathcal{X}_d$ is proper, construct a sequence of nested halfspaces H_n^- such that γ eventually never returns to each halfspace.

Inaccessibility and properness

We can identify regions of the d-plane which are inaccessible by a CL path γ if the initial arc of γ is a subarc of a *proper* ray.

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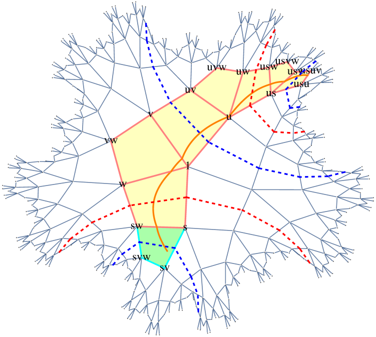
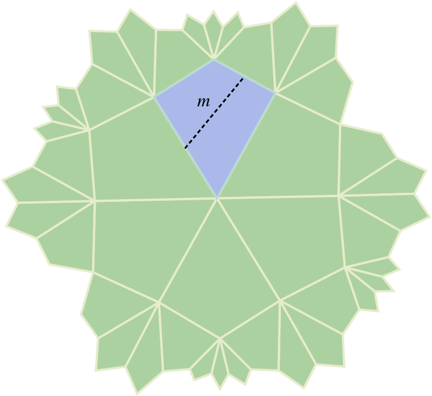


Formally: Construct a sequence of successively osculating hyperplanes, called a *stack*, with respect to which the ray is *properly segmented*.

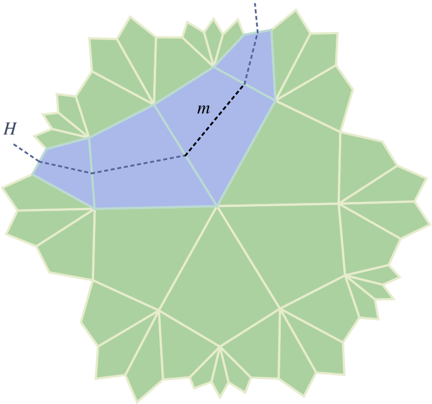
(Definitions to follow.)

Hyperplanes in a square complex



A **midplane** of a 2-cell is a geodesic segment joining the midpoints of its opposite sides.

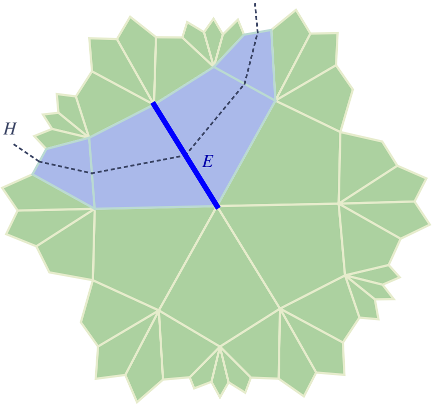
Hyperplanes in a square complex



Two such midplanes m, m' are **equivalent** if their intersection is the midpoint of some edge.

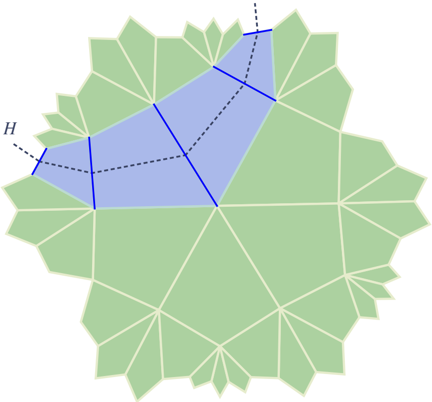
The equivalence class of a midplane under the transitive closure of this relation is a **(1-dimensional) hyperplane**.

Hyperplanes in a square complex



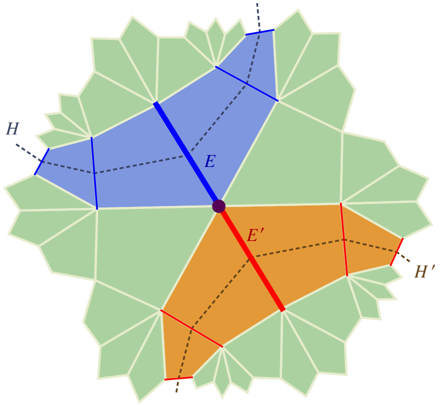
A hyperplane H is **dual** to an edge E if some midplane in H intersects E.

Hyperplanes in a square complex



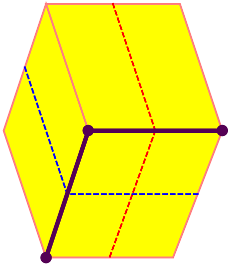
The hyperplanes determine a partition of $\mathrm{Edges}(X)$ into parallelism classes.

Osculating hyperplanes



Let H and H' be hyperplanes of a cube complex X.
We say H and H' **osculate**, and write $H \,)\!(\, H'$,
if there exist adjacent edges E and E' of X such that H is dual to E,
H' is dual to E', and E and E' are not both contained in any 2-cell.

Osculating hyperplanes



Let H and H' be hyperplanes of a cube complex X.
We say H and H' **osculate**, and write $H \,)(\, H'$,
if there exist adjacent edges E and E' of X such that H is dual to E, H' is dual to E', and E and E' are not both contained in any 2-cell.

- *Counterexample:* If the edges E and E' in the definition are adjacent sides of some square cell Q, then H and H' must meet in Q.

Stacks

A **stack** in a cube complex X is a sequence $(H_n)_{n=1}^{\infty}$ of successively osculating hyperplanes, $H_n \;)\!(\; H_{n+1}$.

We would like the hyperplanes in a stack to satisfy two properties:

- Each hyperplane should divide the space (which in our case, is homeomorphic to $\mathbb{R}^2$) into two disjoint halfspaces.
- The hyperplanes should define two sequences of nested halfspaces, a *"backward"* sequence nested from smaller to larger, and a *"forward"* sequence nested from larger to smaller.

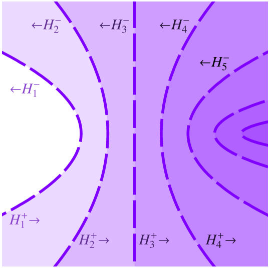
If the hyperplanes of a stack satisfy these two properties, we call it an *oriented* stack.

Stacks

A **stack** in a cube complex X is a sequence $(H_n)_{n=1}^{\infty}$ of successively osculating hyperplanes, $H_n \,)(\, H_{n+1}$.

A stack $(H_n)_{n=1}^{\infty}$ is **oriented** if

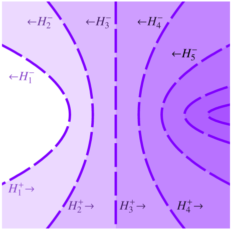
- $X \smallsetminus H_n$ has two connected components for each n, and
- if the components $H_n^{\pm}$ of $X \smallsetminus H_n$ are labeled so that $H_n^- \subset H_{n+1}^-$ and $H_n^+ \supset H_{n+1}^+$ for each n.



Schematic diagram of an oriented stack of hyperplanes (dashed lines)

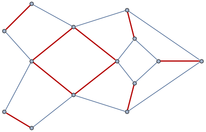
Stacks

But in an arbitrary square complex, osculating hyperplanes need not behave as nicely as our diagram suggests.



We can easily construct square complexes in which

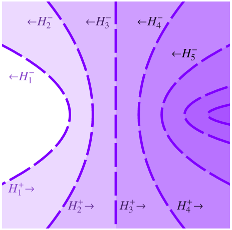
- a hyperplane crosses itself *(edges dual to hyperplane shown in red)*



$\#\{\text{components of complement}\} \neq 2$

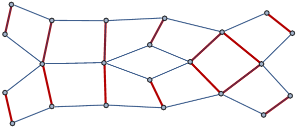
Stacks

But in an arbitrary square complex, osculating hyperplanes need not behave as nicely as our diagram suggests.



We can easily construct square complexes in which

- two osculating hyperplanes intersect *(dual edges highlighted)*



halfspaces not nested

$\mathcal{X}_d$ is an A-special cube complex

Lemma

Let $\mathcal{H} = (H_n)_{n=1}^{\infty}$ be a stack of hyperplanes in $\mathcal{X}_d$ ($d \geqslant 4$). Then $\mathcal{H}$ can be oriented.

Proof (sketch). $\mathcal{X}_d$ is the Davis complex of a RACS, hence a CAT(0) cube complex, hence A-special. Then osculating hyperplanes do not intersect. The complement in $\mathcal{X}_d$ of each hyperplane has two components, and by a connectedness argument, $\mathcal{H}$ can be oriented. □

Showing a ray is proper using osculating hyperplanes

Lemma

A ray $\gamma : [a, \infty) \to \mathcal{X}_d$ is proper if for some

$$a \leqslant t_1 < t_2 < t_3 < \cdots$$

and some oriented stack $(H_n)_{n=1}^{\infty}$ of hyperplanes, we have

$$\gamma\big([t_n, \infty)\big) \subset H_n^+.$$

This Lemma gives a sufficient condition for a ray to be proper.

- The condition is simple, but impractical
- Even for a geodesic path, finding the breakpoints can only be done iteratively
- (A-O-S): The breakpoints of a geodesic have no closed form analytic description
- We need a systematic procedure for building an infinite stack in the *d*-plane
- Rather than focusing on the intersection of the ray with infinitely many hyperplanes, we analyze the square path for the ray, which has an easily described combinatorial structure

Properly segmented square paths

Theorem

A ray γ in $\mathcal{X}_d$ is proper if it has an edgewise square path that is properly segmented with respect to some oriented stack of hyperplanes.

We now define *properly segmented* for a square path, rather than for a ray…

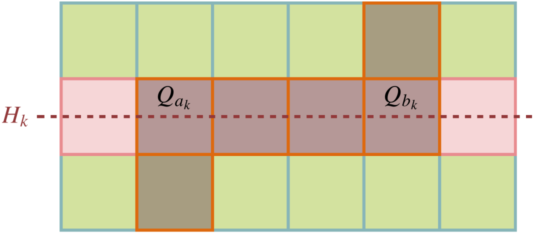
Properly segmented square paths

Theorem

A ray γ in $\mathcal{X}_d$ is proper if it has an edgewise square path that is properly segmented with respect to some oriented stack of hyperplanes.

An edgewise square path $(Q_n)_{n=1}^{\infty}$ is **properly segmented** by an oriented stack $(H_k)_{k=1}^{\infty}$ if there exist $1 \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots$ such that for each $k \in \mathbb{N}$,

(1) $\bigcup_{n=a_k}^{b_k} Q_n \subset \text{Carrier}(H_k)$,



The **carrier** of a subset A of a cube complex X is the smallest subcomplex of X containing A.

Properly segmented square paths

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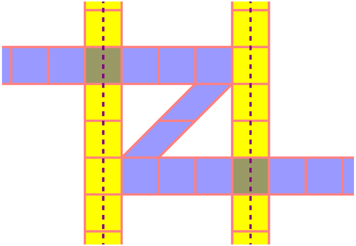
(1) $\bigcup_{n=a_k}^{b_k} Q_n \subset \mathrm{Carrier}(H_k)$,

(2) $\bigcup_{n=b_k+1}^{a_{k+1}-1} Q_n \subset X \setminus (H_k \cup H_{k+1})$,



Properly segmented square paths

Theorem

A ray γ in $\mathcal{X}_d$ is proper if it has an edgewise square path that is properly segmented *with respect to some oriented stack of hyperplanes.*

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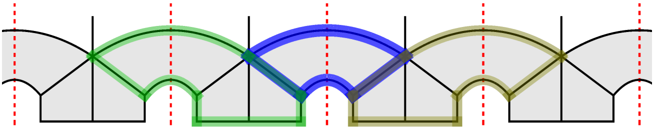
(2) $\bigcup_{n=b_k+1}^{a_{k+1}-1} Q_n \subset X \setminus (H_k \cup H_{k+1})$, and

(3) $\bigcup_{n=a_{k-1}}^{a_k-1} Q_n$ and $\bigcup_{n=b_k+1}^{b_{k+1}} Q_n$ meet distinct components of $\left(\bigcup_{n=a_k}^{b_k} Q_n\right) \setminus H_k$.



Properly segmented square paths

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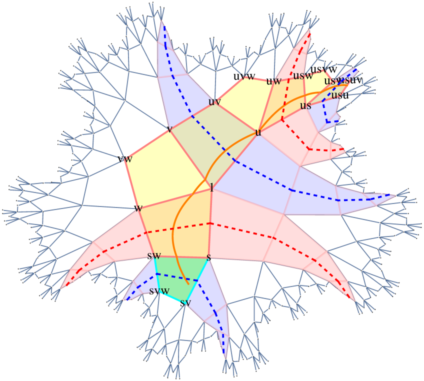


Given a square path $\mathcal{Q}$ for a ray of constant curvature, how do we build a stack of hyperplanes with respect to which $\mathcal{Q}$ is properly segmented?

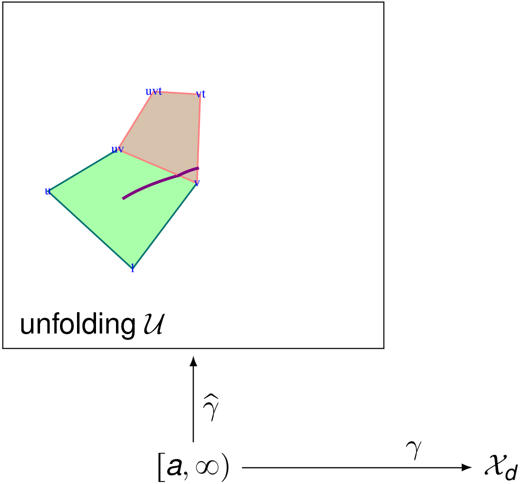
Unfolding and refolding

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.

$$[a, \infty) \xrightarrow{\quad\gamma\quad} \mathcal{X}_d$$

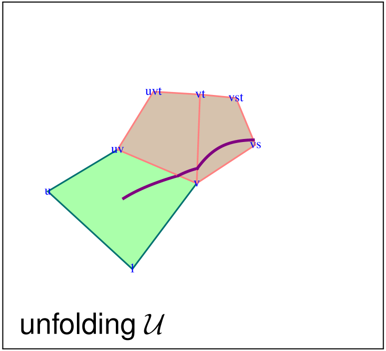
Unfolding and refolding

Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$



Unfolding and refolding

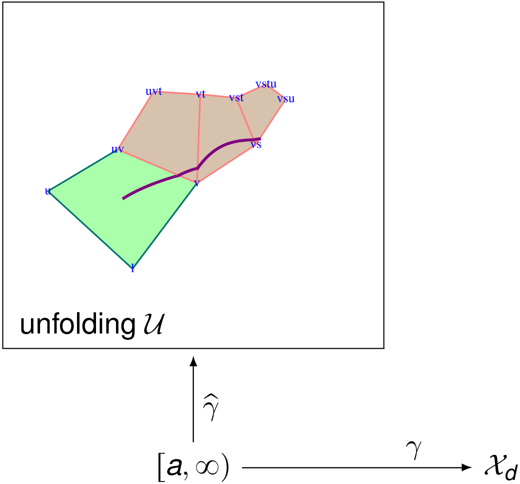
Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$



$$\begin{array}{ccc} \mathcal{U} & & \\ \uparrow \hat{\gamma} & & \\ [a, \infty) & \xrightarrow{\gamma} & \mathcal{X}_d \end{array}$$

Unfolding and refolding

Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$



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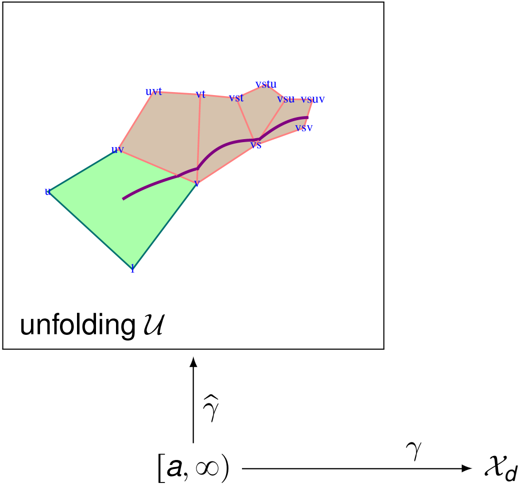

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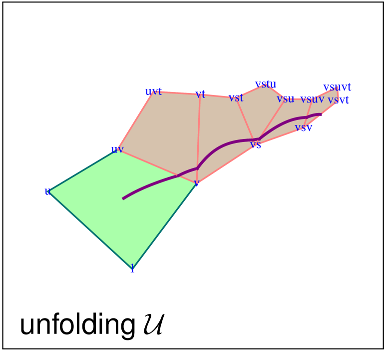
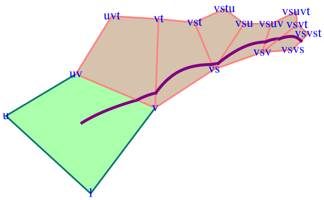


$$
\begin{array}{ccc}
\mathcal{U} & & \\
\uparrow \hat{\gamma} & & \\
[a, \infty) & \xrightarrow{\gamma} & \mathcal{X}_d
\end{array}
$$

Unfolding and refolding

Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$

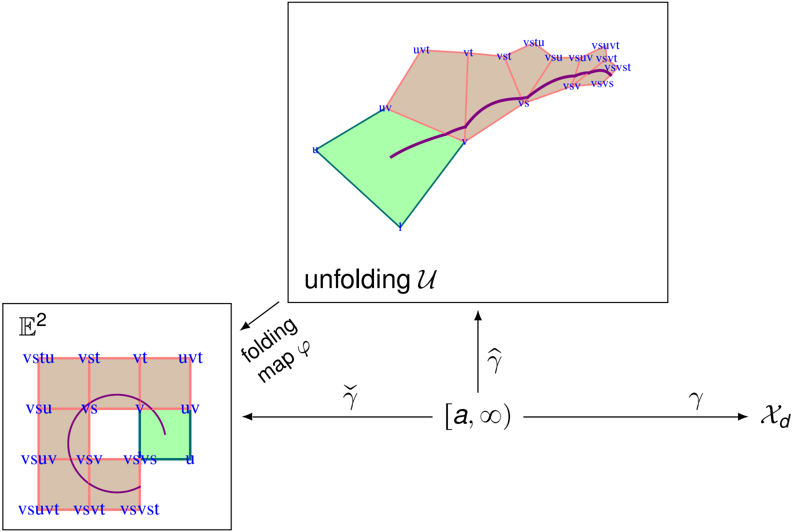


unfolding $\mathcal{U}$

$$\begin{array}{ccc} \mathcal{U} & & \\ \uparrow \hat{\gamma} & & \\ [a,\infty) & \xrightarrow{\gamma} & \mathcal{X}_d \end{array}$$

Unfolding and refolding

Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$



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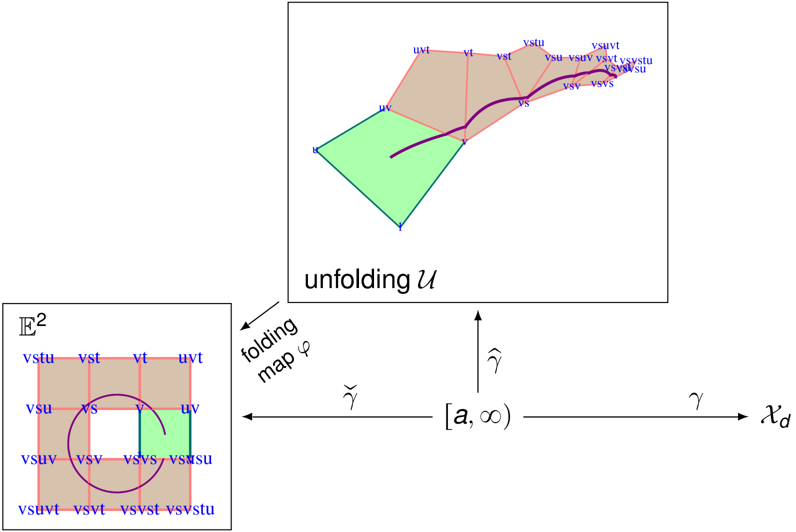
Unfolding complex $\mathcal{U}$ and folding map $\mathcal{U} \to \mathbb{E}^2$



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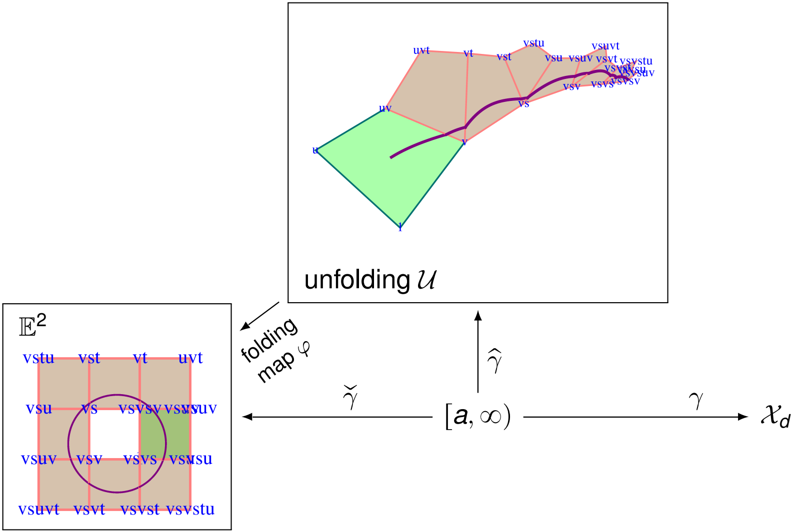
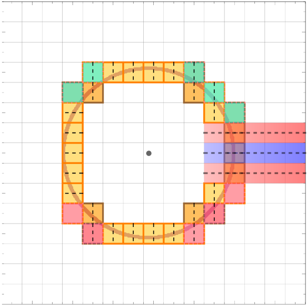


Finite stacks in $\mathbb{E}^2$

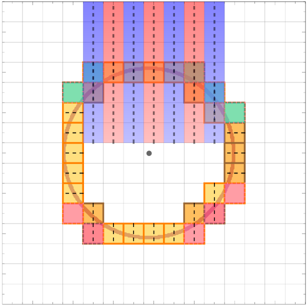
- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.



After identifying these four finite stacks in $\mathbb{E}^2$, we have enough information to construct an infinite stack in $\mathcal{X}_d$ with respect to which the square path of the original ray γ is properly segmented, and it follows that γ is proper.

Finite stacks in $\mathbb{E}^2$

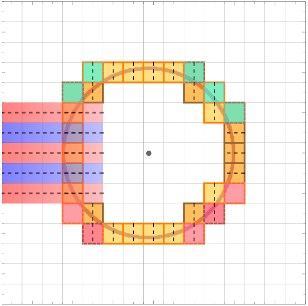
- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.



We will transfer each of the four finite stacks into $\mathcal{X}_d$.

Finite stacks in $\mathbb{E}^2$

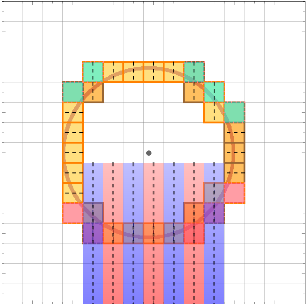
- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.



When transferring a stack from $\mathbb{E}^2$ to $\mathcal{X}_d$, choices must be made.

Finite stacks in $\mathbb{E}^2$

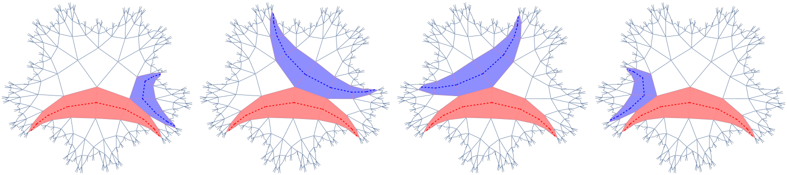
- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.



The carriers of a pair of osculating hyperplanes of $\mathbb{E}^2$ meet along infinitely many pairs of incident 2-cells...

Transferring a stack

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.
- Transfer each stack in $\mathbb{E}^2$ to $\mathcal{X}_d$.

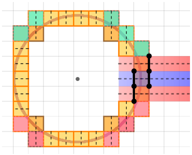


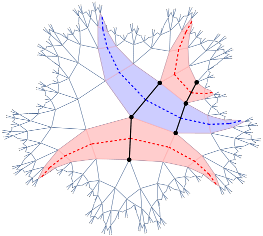
. . . and each selection of such a pair determines a different pair of hyperplanes in $\mathcal{X}_d$.

We keep track of our choices using what we call a *scaffold*.

Transferring a stack

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.
- Transfer each stack in $\mathbb{E}^2$ to $\mathcal{X}_d$.



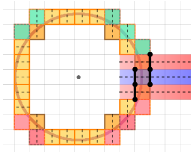


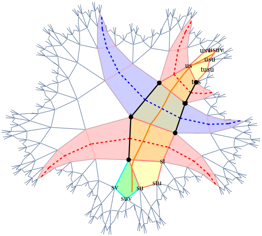
A *scaffold* is the minimal data needed to carry out the transfer of a stack between two square complexes in a controlled way.

It consists of a sequence of pairs of adjacent edges, respectively dual to each pair of successive hyperplanes in a stack.

Transferring a stack

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.
- Transfer each stack in $\mathbb{E}^2$ to $\mathcal{X}_d$.

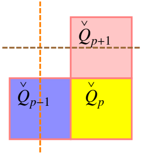


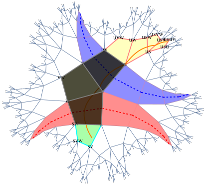


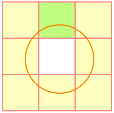
The original ray in $\mathcal{X}_d$ is properly segmented by the resulting finite stack.

Infinite stack in $\mathcal{X}_d$

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.
- Transfer each stack in $\mathbb{E}^2$ to $\mathcal{X}_d$.
- Assemble the finite stacks in $\mathcal{X}_d$ into an infinite sequence of successively osculating hyperplanes.





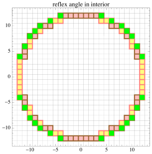


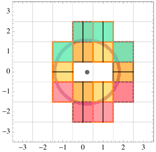
Finally, we assemble the sequence of finite stacks in the d-plane, obtained from the four finite stacks in $\mathbb{E}^2$, into a single infinite stack with respect to which the given square path in $\mathcal{X}_d$ is properly segmented.

Using the nonpositive curvature of $\mathcal{X}_d$, we can assemble the finite stacks into a single infinite stack if…

Infinite stack in $\mathcal{X}_d$

- Transfer a square path $\mathcal{Q} = (Q_k)_{k=1}^{\infty}$ for γ in $\mathcal{X}_d$ to $\mathbb{E}^2$ by continuation, keeping track of cell structure.
- The resulting ray $\check{\gamma}$ is a parametrized circle. Subdivide $\check{\gamma}$ into four arcs, each of which is properly segmented by a finite stack in $\mathbb{E}^2$.
- Transfer each stack in $\mathbb{E}^2$ to $\mathcal{X}_d$.
- Assemble the finite stacks in $\mathcal{X}_d$ into an infinite sequence of successively osculating hyperplanes.






```
isElbowJoint[squareCenter_, indexInSquarePath_] :=
  (angle @@ neighborCenters[squareCenter, indexInSquarePath]) == π / 2;
```

. . . we can find four suitable “reflex angles” in the interior component of the boundary of the square path in $\mathbb{E}^2$.

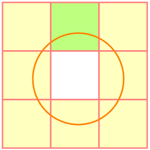
This can always be done if the carrier of the circle in $\mathbb{E}^2$ is an annulus.

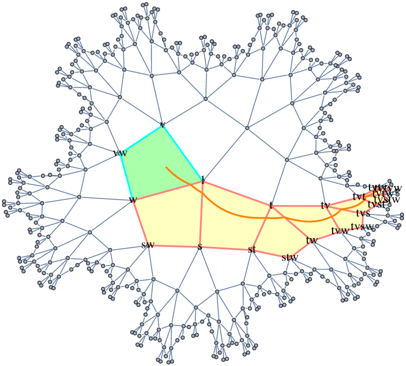
Sufficient condition for a ray in $\mathcal{X}_d^*$ of constant curvature to be proper

Annulus Condition

Let $\gamma : [a, \infty) \to \mathcal{X}_d^* = \mathcal{X}_d \setminus \mathrm{Vert}(\mathcal{X}_d)$ *($d \geqslant 5$) be a curve of constant curvature* $\kappa > 0$*. Let* $\mathcal{U}$ *be an unfolding of a locally monotone edgewise square path* $\mathcal{Q}$ *in* $\mathcal{X}_d$ *for* γ*. Let* $\varphi : \mathcal{U} \to \mathbb{E}^2$ *be a cellular local isometry.*

If $\mathrm{Image}\,\varphi \overset{homeo}{\approx} S^1 \times I$*, then* γ *is a proper ray.*



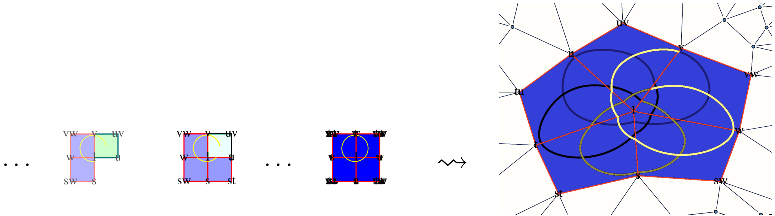


Characterization of curves of constant curvature $\kappa > 0$

Small Block Condition

Let $\gamma : [a, \infty) \to \mathcal{X}_d^*$ *($d \geqslant 5$) be a curve of constant curvature* $\kappa > 0$. *Let* $\mathcal{U}$ *be an unfolding of a locally monotone edgewise square path* $\mathcal{Q}$ *in* $\mathcal{X}_d$ *for* γ. *Let* $\varphi : \mathcal{U} \to \mathbb{E}^2$ *be a cellular local isometry.*

If $\text{Image}\,\varphi \overset{isom}{\cong} [-1,1] \times [-1,1]$, *then* $\text{Image}\,\gamma$ *is either an embedded circle, or a rose curve made up of* $M = \text{lcm}\{4, d\}$ *arcs.*



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