Curvature without calculus: An introduction to comparison geometry

Julie Carmela La Corte

University of Wisconsin–Milwaukee

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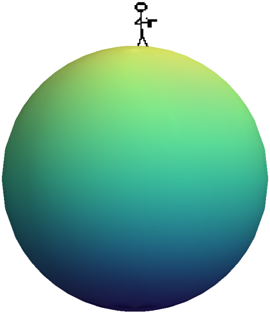
Thought experiment:

The curvature of the Earth's surface at a point

Imagine a person standing on the Earth's surface and firing a magic bullet that is not affected by gravity.



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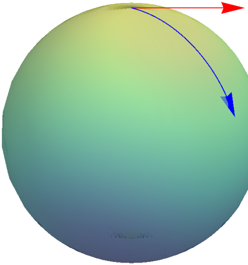

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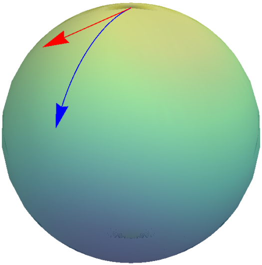

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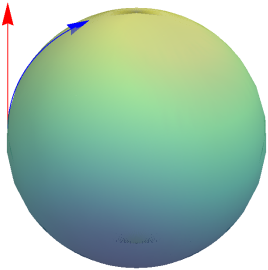
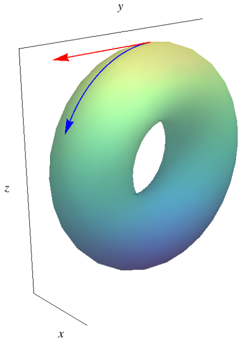


The Earth's surface curves away from the bullet's trajectory, no matter what direction the shooter faces, and no matter where she stands.

Thought experiment:

The curvature of the torus at a point

The situation would be different if the surface of our planet was shaped like a torus.



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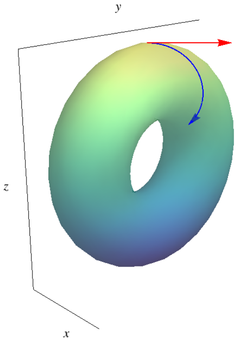

A shooter standing at certain points of the torus would observe Earth-like curvature.

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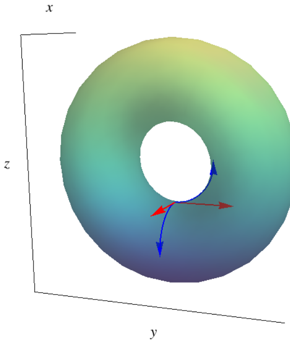
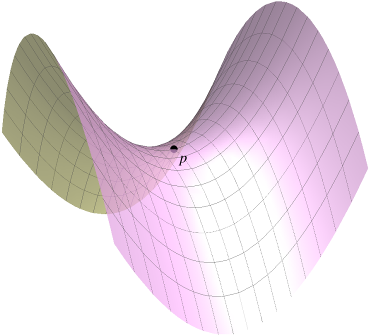


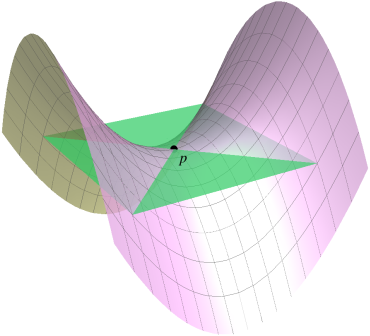
But at a point on the inner rim of the torus, the shooter would see the torus curving away from the bullet's trajectory in some directions, and toward it in other directions.

Normal vector to a surface



A neighborhood of a point p on the inner rim of the torus resembles the *saddle surface*.

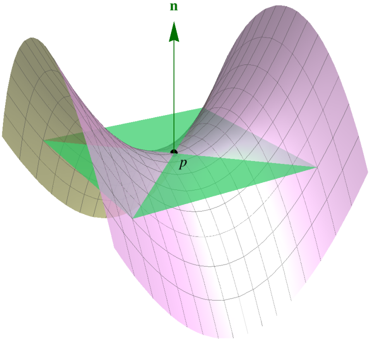
Normal vector to a surface



A neighborhood of a point *p* on the inner rim of the torus resembles the *saddle surface*.

Let's draw a plane tangent to this surface at *p*.

Normal vector to a surface

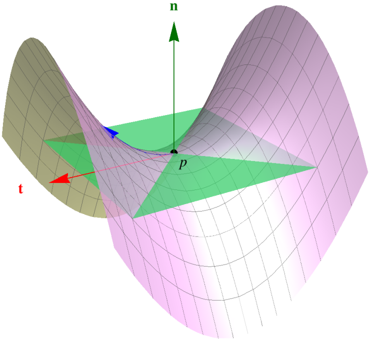


A neighborhood of a point *p* on the inner rim of the torus resembles the *saddle surface*.

Let's draw a plane tangent to this surface at *p*.

Label one of the directions perpendicular to the tangent plane as **n**. This direction will be called "up."

Negative curvature at a point on a surface



For certain initial directions **t**, curves along the saddle surface that start at *p* bend “up,” toward **n**.

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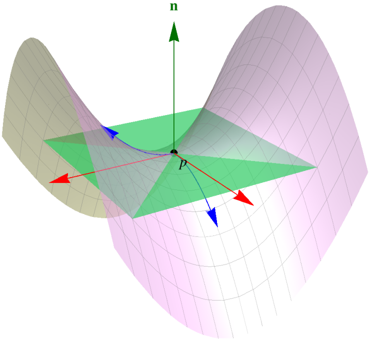

For certain initial directions **t**, curves along the saddle surface that start at *p* bend “up,” toward **n**.

For other choices of **t**, they bend “down,” away from **n**.

We say that such a point *p* is **hyperbolic**, and that the surface is **negatively curved** at *p*.

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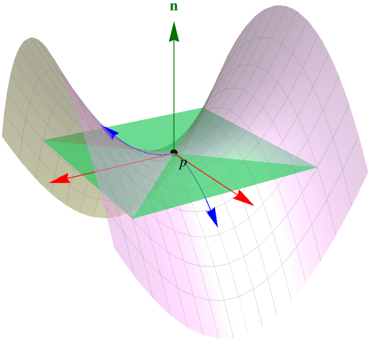

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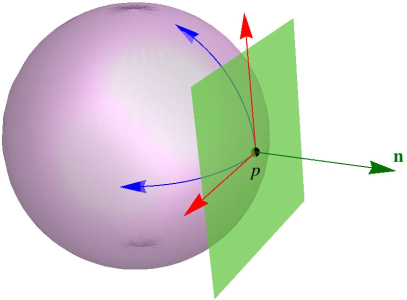
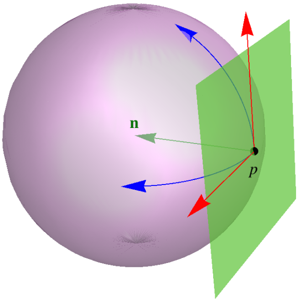
Positive curvature at a point on a surface

A surface is **positively curved** at a point p, and p is an **elliptic** point, if:

- for any choice of initial direction **t**, all curves along the surface starting at p bend toward **n**, or all bend away from **n**.



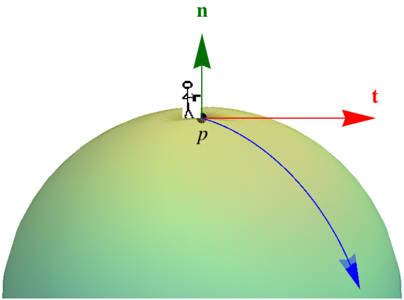


Normal curvature

In calculus, we learn that the **normal curvature**

$$\kappa = \kappa(\mathbf{t})$$

measures how quickly a surface appears to bend in a given direction **t** when standing at a point *p* on the surface.



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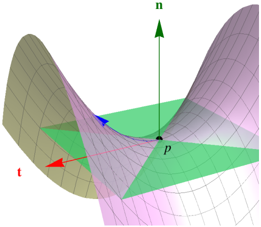
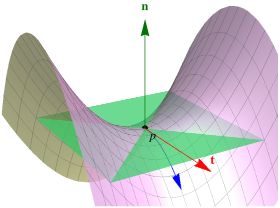
- If $\kappa > 0$, the surface bends in the direction of **n**.
- If $\kappa < 0$, the surface bends in the opposite direction.



$\kappa(\mathbf{t}) > 0$



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Normal curvature

Computing the normal curvature κ isn't hard, but it can be tedious. . .

Write $\mathbf{t} = \begin{bmatrix} du \\ dv \end{bmatrix}$. Then

$$\kappa\left(\begin{bmatrix} du \\ dv \end{bmatrix}\right) = -\frac{\left(\frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial u}\right) du^2 + \left(\frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial v} + \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial u}\right) du\, dv + \left(\frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial v}\right) dv^2}{\left(\frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial u}\right) du^2 + 2\left(\frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v}\right) du\, dv + \left(\frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial v}\right) dv^2},$$

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Curvature in n-dimensional spaces ($n \geq 3$)

It makes sense to speak of the curvature of higher-dimensional geometric figures, too, although the mathematics is much more difficult.

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☞ To a topologist, the sphere, the torus, and the saddle surface are 2-*dimensional* spaces.

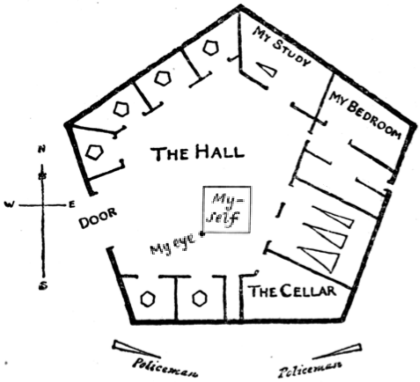

Image from E.A. Abbott, *Flatland: A Romance of Many Dimensions* (1884)

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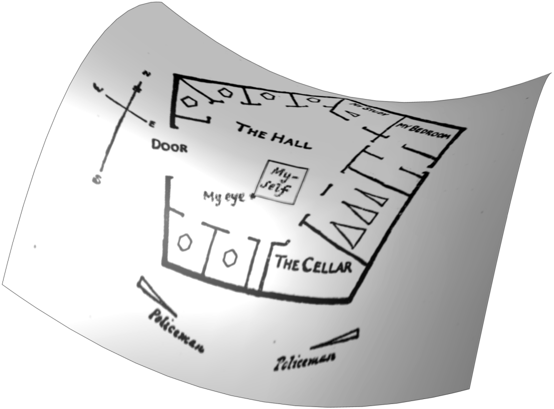

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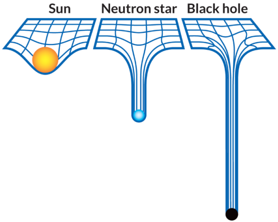
The theory of general relativity tells us that our universe, which we experience as being 3-dimensional, is “curved” in the vicinity of massive objects.



The curvature of n-dimensional shapes for $n \geq 3$ is studied in the branch of mathematics known as **differential geometry**.

Image from *Science News*, May 16, 2014, http://www.sciencenews.org/article/mysterious-boundary

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☞ As we look at spaces of higher and higher dimension, we need more and more calculations to characterize their curvature.

Why comparison geometry?

Comparison geometry is an alternative method for describing the curvature of a space.

- *It's simple:*
  No calculus (or differential geometry) is required.
- *It scales well:*
  Exactly the same amount of work is required to describe the curvature of a space of *any* dimension.
- *It's purely geometric:*
  Comparison geometry is based on triangles and distances.
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Metric spaces

The first three axioms define *distance.*

- Given any geometric figure (any set, even), we can define what the distance between two points is.

A **distance function** on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following axioms.

- $d(x, x) = 0$ for all x in X.
- $d(x, y) = d(y, x)$ for all x and y in X.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X.
  You can't take a shortcut from x to z by going through y.

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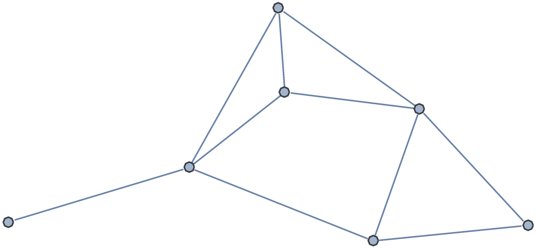
Examples of metric spaces:

- Graphs



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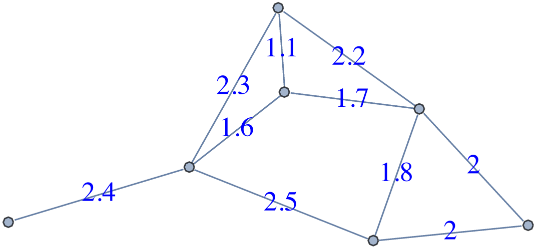

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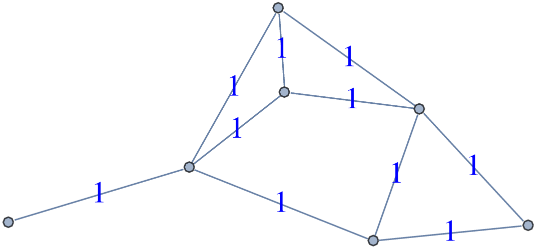

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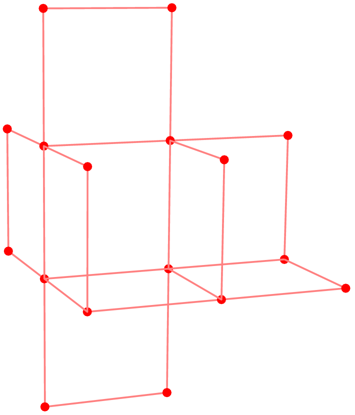
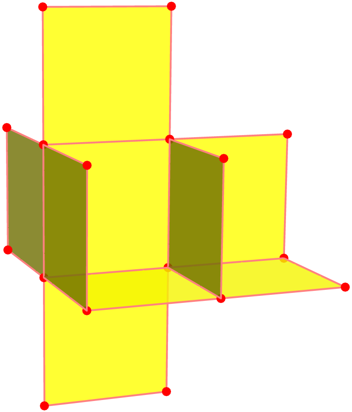


Metric spaces

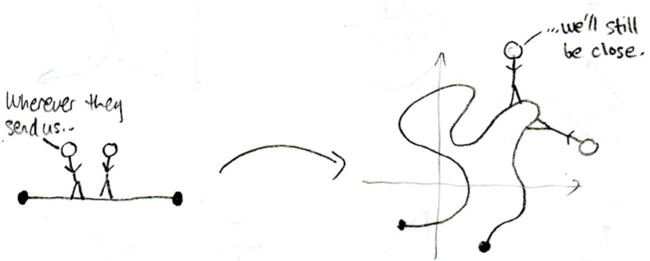
Examples of metric spaces:

- Square complexes



Paths

A **path** in a space X is a continuous function from the unit interval to X.
That is, nearby points are sent to nearby points.



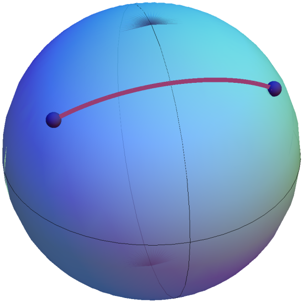
A shortest path between two points x and y is called a **geodesic path** between x and y.

Geodesic spaces

4th axiom:
A metric space X is called a **geodesic space** if there is a(t least one) shortest path between any two points in X.

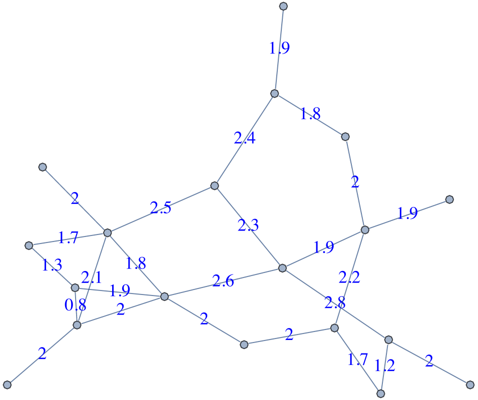
Example of a geodesic space:

- Sphere

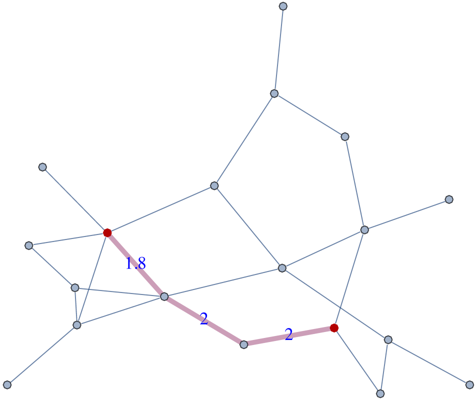


Geodesic paths

Example of a geodesic space:



Geodesic triangles



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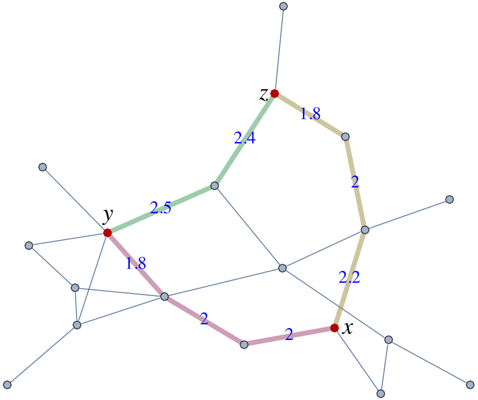
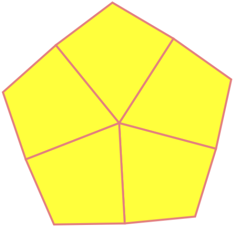


A **geodesic triangle** Δxyz is a “triangle” whose sides are geodesic paths joining x, y, and z in pairs.

Comparison triangles

We can describe the curvature of a space using only triangles.

Suppose Δxyz is a geodesic triangle in a metric space. A **comparison triangle** $\Delta x'y'z'$ is a triangle in the (Euclidean) plane with corresponding sides equal in length.



See cardboard model. . .

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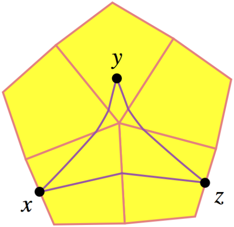

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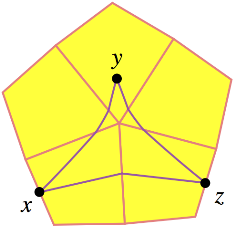
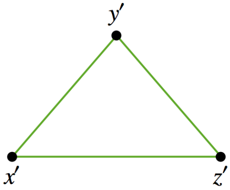




Nonpositive curvature: The "thin triangles" definition

A geodesic triangle Δxyz is **thin** if the distance between any two points on Δxyz is no larger than the distance between the corresponding points on a comparison triangle:

$$d(p, q) \leq d(p', q').$$

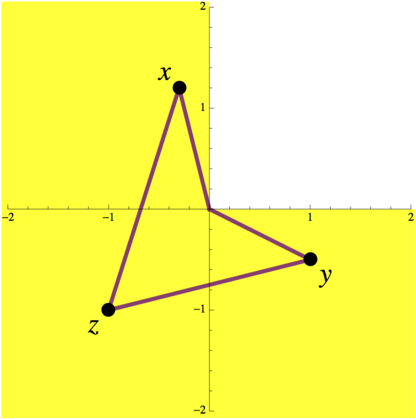
We say that a geodesic space is **nonpositively curved (NPC)** at a point w if all geodesic triangles sufficiently near w are thin.

In a **nonpositively curved** space, all geodesic triangles are thin.

Nonpositively curved spaces

Examples of nonpositively curved spaces:

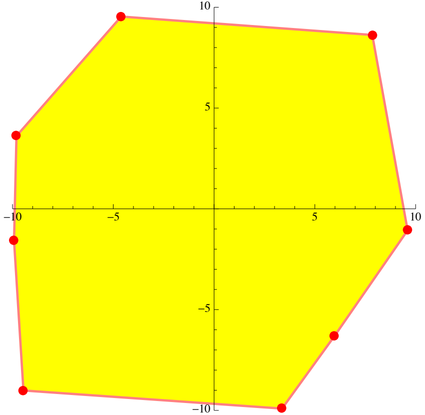
- The Cartesian plane with Quadrant I removed



Nonpositively curved spaces

Examples of nonpositively curved spaces:

- Convex subsets of the plane



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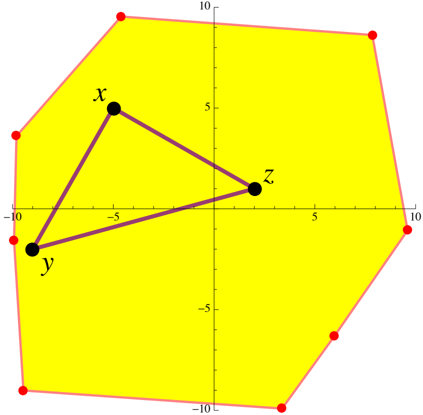
Examples of nonpositively curved spaces:

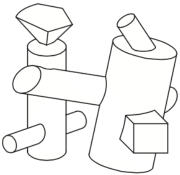
- Convex subsets of the plane



Nonpositively curved spaces

Examples of nonpositively curved spaces:

- Unions of convex sets glued along convex subsets



- Cylindrically deleted cubes

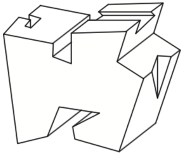
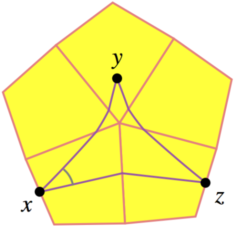


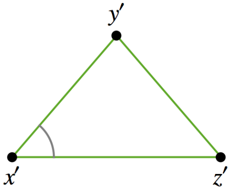
Image from S. Alexander, R. Bishop, R. Ghrist, “Pursuit and evasion in non-convex domains of arbitrary dimension” (2006)

Nonpositive curvature: The "pointy angles" definition

An alternative way to characterize nonpositively curved spaces:

A geodesic triangle is **pointy** if the angle between any two of its sides is no larger than the corresponding angle of a comparison triangle.



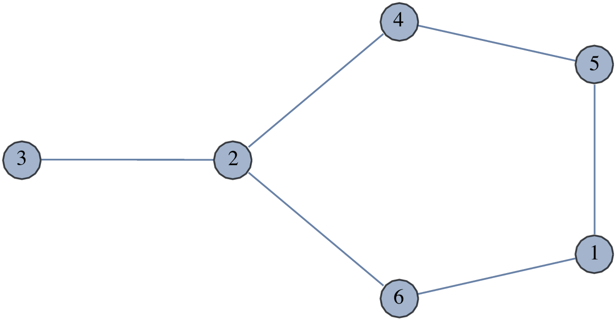


A geodesic space is **nonpositively curved (NPC)** near a point *w* if all triangles sufficiently near *w* are pointy.

Nonpositive curvature: The “pointy angles” definition

More examples:

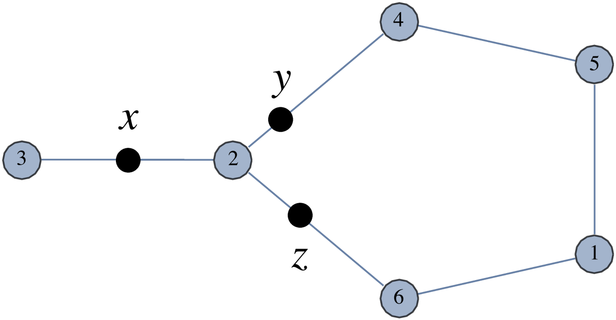
- A graph is nonpositively curved at every point.



Nonpositive curvature: The "pointy angles" definition

More examples:

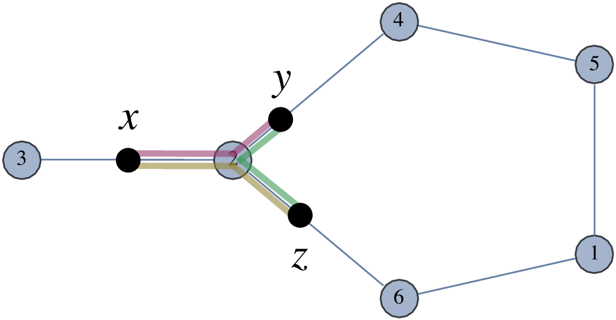
- A graph is nonpositively curved at every point.



Nonpositive curvature: The "pointy angles" definition

More examples:

- A graph is nonpositively curved at every point.



Gromov's Link Condition

Consider a surface made up of n equilateral triangles arranged cyclically around a central vertex v.

When $n = 5$, the surface contains "fat" triangles.

Do any of the surfaces we saw today contain fat triangles?

Gromov's Link Condition.

- A surface made up of n equilateral triangles arranged cyclically around a central vertex v is nonpositively curved at v if and only if. . .

What is comparison geometry good for?

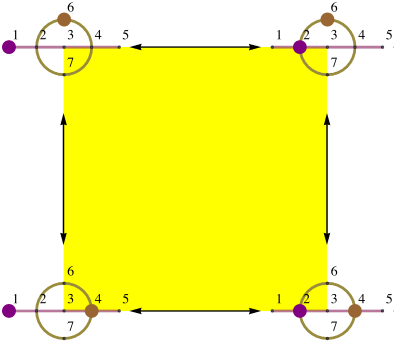
Comparison geometry has practical applications to robotics, chemical engineering, biology. . .

For example, many robotic systems can be modeled by nonpositively curved cube complexes.

Two robots moving along tracks on a factory floor

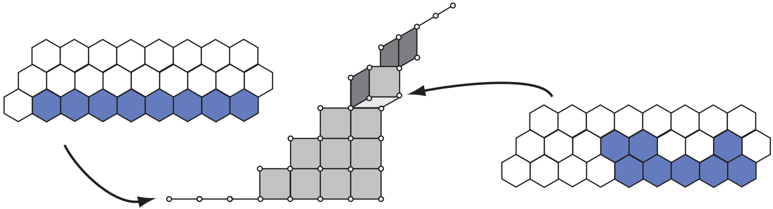
State complex

Every point in the **state complex** corresponds to a possible configuration of the system.



Shape-changing robots

Every point in the **state complex** corresponds to a possible configuration of the system.



State complex for a metamorphic robotic system composed of pivoting hexagonal tiles

Image from R. Ghrist and V. Peterson, “The geometry and topology of reconfiguration” (2007)

Planning a change of state

Paths in the state complex correspond to reconfiguration strategies.

- All computer-generated pictures and animations created in Mathematica except where noted.