Arrow's Impossibility Theorem, or: How to rig an election

Julie Carmela La Corte

University of Wisconsin–Milwaukee

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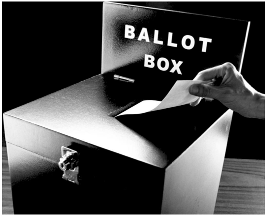
The Election Process



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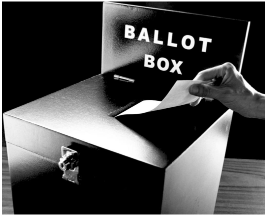

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Count the ballots, and decide the outcome of the election in a *consistently fair* manner.

Surely there must be a reasonable way to accomplish this.

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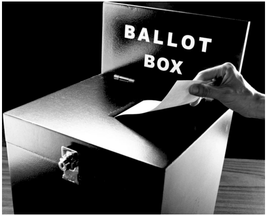

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- A method for deciding the outcome of an election is called a **voting scheme**.
- Arrow stated minimum requirements, called **fairness criteria**, which define what it means for a voting scheme to be "consistently fair."
- Today we'll look at some different voting schemes, and introduce Arrow's fairness criteria as we go.

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Preference Ballots

In all our elections, each voter will be asked to fill out a **preference ballot** in which she ranks all the candidates in order of preference.

Ex. Voters were asked,

Who is the greatest R&B singer of all time?

Their choices were Al Green, Beyoncé, Ray Charles, and Diana Ross.

Sample preference ballot:

Ballot	
1st	*Diana Ross*
2nd	*Ray Charles*
3rd	*Beyoncé*
4th	*Al Green*

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Who is the greatest R&B singer of all time?

Their choices were **A**l Green, **B**eyoncé, Ray **C**harles, and **D**iana Ross.

<u>Sample preference ballot</u>:

Ballot	
1st	**D**
2nd	**C**
3rd	**B**
4th	**A**

Preference Schedules

Ex. *A*lisha, *B*oris, *C*armen, and *D*ave are running for president of the Math Appreciation Society at Tasmania State University.

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*B*
2nd	*D*
3rd	*C*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

Ballot	
1st	*B*
2nd	*D*
3rd	*C*
4th	*A*

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*B*
2nd	*D*
3rd	*C*
4th	*A*

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

Ballot	
1st	*D*
2nd	*C*
3rd	*B*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*C*
2nd	*B*
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Ballot	
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2nd	*B*
3rd	*C*
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Ballot	
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Ballot	
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2nd	*C*
3rd	*B*
4th	*A*

Ballot	
1st	*C*
2nd	*B*
3rd	*D*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*D*
2nd	*C*
3rd	*B*
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Ballot	
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Ballot	
1st	*C*
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3rd	*D*
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Ballot	
1st	*C*
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Ballot	
1st	*D*
2nd	*C*
3rd	*B*
4th	*A*

Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Ballot	
1st	*D*
2nd	*C*
3rd	*B*
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Ballot	
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Ballot	
1st	*A*
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Ballot	
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2nd	*C*
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Ballot	
1st	*A*
2nd	*B*
3rd	*C*
4th	*D*

Preference Schedules

Arrow's Impossibility Theorem

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Introduction
- Arrow's Impossibility Theorem
- Preference Schedules

Voting schemes
- Plurality Method
- Borda Count Method
- The Method of Pairwise Comparisons
- Instant runoff voting

Group work

Ex. *A*lisha, *B*oris, *C*armen, and *D*ave are running for president of the Math Appreciation Society at Tasmania State University.

Ballot		Ballot		Ballot		Ballot		Ballot		Ballot		Ballot	
1st	*A*	1st	*B*	1st	*A*	1st	*C*	1st	*B*	1st	*C*	1st	*A*
2nd	*B*	2nd	*D*	2nd	*B*	2nd	*B*	2nd	*D*	2nd	*B*	2nd	*B*
3rd	*C*	3rd	*C*	3rd	*C*	3rd	*D*	3rd	*C*	3rd	*D*	3rd	*C*
4th	*D*	4th	*A*	4th	*D*	4th	*A*	4th	*A*	4th	*A*	4th	*D*

Ballot		Ballot		Ballot		Ballot		Ballot		Ballot		Ballot	
1st	*B*	1st	*C*	1st	*A*	1st	*C*	1st	*D*	1st	*A*	1st	*A*
2nd	*D*	2nd	*B*	2nd	*B*	2nd	*B*	2nd	*C*	2nd	*B*	2nd	*B*
3rd	*C*	3rd	*D*	3rd	*C*	3rd	*D*	3rd	*B*	3rd	*C*	3rd	*C*
4th	*A*	4th	*A*	4th	*D*	4th	*A*	4th	*A*	4th	*D*	4th	*D*

Ballot		Ballot		Ballot		Ballot		Ballot		Ballot		Ballot	
1st	*C*	1st	*A*	1st	*C*	1st	*D*	1st	*C*	1st	*A*	1st	*D*
2nd	*B*	2nd	*B*	2nd	*B*	2nd	*C*	2nd	*B*	2nd	*B*	2nd	*C*
3rd	*D*	3rd	*C*	3rd	*D*	3rd	*B*	3rd	*D*	3rd	*C*	3rd	*B*
4th	*A*	4th	*D*	4th	*A*	4th	*A*	4th	*A*	4th	*D*	4th	*A*

Ballot		Ballot		Ballot		Ballot		Ballot		Ballot		Ballot	
1st	*D*	1st	*C*	1st	*C*	1st	*D*	1st	*A*	1st	*D*	1st	*C*
2nd	*C*	2nd	*B*	2nd	*B*	2nd	*C*	2nd	*B*	2nd	*C*	2nd	*B*
3rd	*B*	3rd	*D*	3rd	*D*	3rd	*B*	3rd	*C*	3rd	*B*	3rd	*D*
4th	*A*	4th	*A*	4th	*A*	4th	*A*	4th	*D*	4th	*A*	4th	*A*

Ballot		Ballot		Ballot		Ballot		Ballot		Ballot		Ballot	
1st	*A*	1st	*D*	1st	*B*	1st	*A*	1st	*C*	1st	*A*	1st	*A*
2nd	*B*	2nd	*C*	2nd	*D*	2nd	*B*	2nd	*D*	2nd	*B*	2nd	*B*
3rd	*C*	3rd	*B*	3rd	*C*	3rd	*C*	3rd	*B*	3rd	*C*	3rd	*C*
4th	*D*	4th	*A*	4th	*A*	4th	*D*	4th	*A*	4th	*D*	4th	*D*

Ballot		Ballot	
1st	*D*	1st	*A*
2nd	*C*	2nd	*B*
3rd	*B*	3rd	*C*
4th	*A*	4th	*D*

Preference Schedules

There are only a handful of possible ways to fill out the ballot. If we stack identical ballots together, we see that there were only five different ballots in the election.

We can summarize the ballot information with a **preference schedule**, a table which shows how many of each type of ballot there were:

number of voters:	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Henceforth, we will give all elections in the form of a preference schedule.

Plurality Method

One way to decide the winner of the Math Appreciation Society election is to declare that the winner is the candidate with the most 1st choice votes.

We call this the **plurality method** for deciding the election.

number of voters:	14	10	8	4	1
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Is there any problem with awarding this election to Alisha?

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Majority Candidates

A **majority** is more than 50% of the votes.

1824 presidential election

candidate	percent of electoral votes won
Andrew Jackson	37.9%
John Quincy Adams	32.2%
William Crawford	15.7%
Henry Clay	14.2%

62.1%

"The election of Paul LePage with 38% of the vote means Maine's next governor won't take office with the support of the majority of voters—a situation that has occurred in six of the last seven gubernatorial elections."
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Insincere Voting

number of voters:	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*

There is another problem with the plurality method.

Insincere Voting

number of voters:	14	10	8	4	1
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2*nd choice*	*B*	*B*	*C*	*D*	*D*
3*rd choice*	*C*	*D*	*B*	*C*	*B*
4*th choice*	*D*	*A*	*A*	*A*	*A*

Realizing that their candidate has no chance of winning, *B*oris's supporters (pink) decide that they would be "wasting their vote," and cast their votes for *C*armen instead.

Insincere Voting

number of voters:	14	10	8	4	1
1st choice	*A*	*C*	*D*	*C*	*C*

We see that, if the plurality method is used to decide the winner, voters are pressured to vote for one of only two candidates.

Duverger's Law. The plurality method necessarily leads to a two-party system, given enough time.

Is this the two-party system unfair?

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We say a voting scheme **violates** the Majority Criterion if it is *possible* for a majority candidate to lose the election.

Q: Does the Plurality Method violate the Majority Criterion?

The Borda Count Method

Do you know of any elections in which 2nd-choice, 3rd-choice, etc., votes actually count?

The Borda Count Method

Do you know of any elections in which 2nd-choice, 3rd-choice, etc., votes actually count?

The recipient of the MVP Award in Major League Baseball is chosen by the Baseball Writers Association of America.

- Each member of the Association ranks the candidates from 1st choice to last choice.
- A last-choice ranking is worth 1 point. A next-to-last-choice ranking is worth 2 points, and so on.

$= 4$ points

$= 3$ points

$= 2$ points

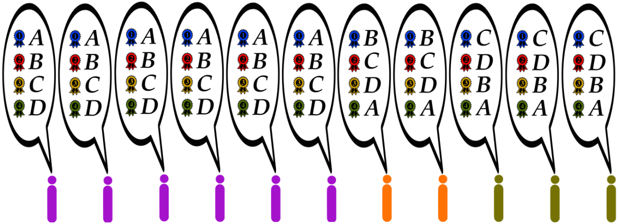
$= 1$ point

The winner is the candidate with the most *points*.

We call this scheme the **Borda count method**.

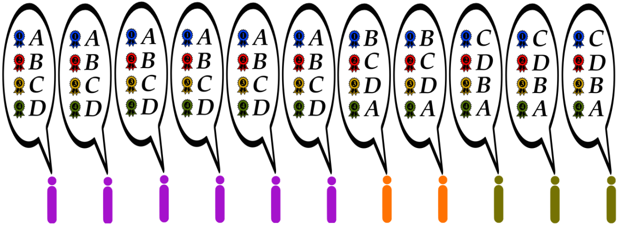
The Borda Count Method: Sample election

Suppose that 11 voters choose the recipient of the MVP Award from the four candidates *A*bbott, *B*utler, *C*astillo, and *D*avis.



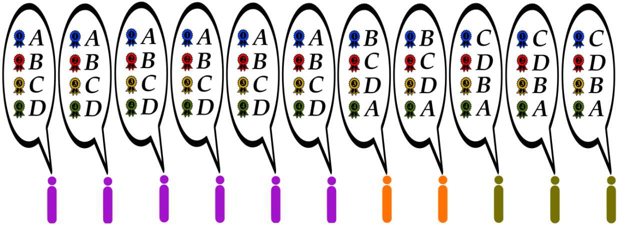
number of voters:	6	2	3
1st choice	*A*	*B*	*C*
2nd choice	*B*	*C*	*D*
3rd choice	*C*	*D*	*B*
4th choice	*D*	*A*	*A*

The Borda Count Method: Sample election



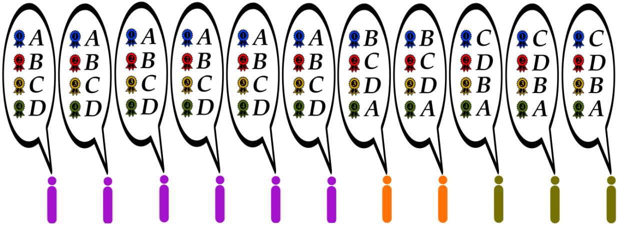
	6	2	3
1st choice	A: $4 \times 6 = 24$	B	C
2nd choice	B	C	D
3rd choice	C	D	B
4th choice	D	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

The Borda Count Method: Sample election



	6	2	3
1st choice	A: 4 pts	B: 4 pts	C: 4 pts
2nd choice	B: 3 pts	C: 3 pts	D: 3 pts
3rd choice	C: 2 pts	D: 2 pts	B: 2 pts
4th choice	D: 1 pt	A: 1 pt	A: 1 pt

The Borda Count Method: Sample election



	6	2	3
1st choice	A: $4 \times 6 = 24$	B: $4 \times 2 = 8$	C: $4 \times 3 = 12$
2nd choice	B: $3 \times 6 = 18$	C: $3 \times 2 = 6$	D: $3 \times 3 = 9$
3rd choice	C: $2 \times 6 = 12$	D: $2 \times 2 = 4$	B: $2 \times 3 = 6$
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

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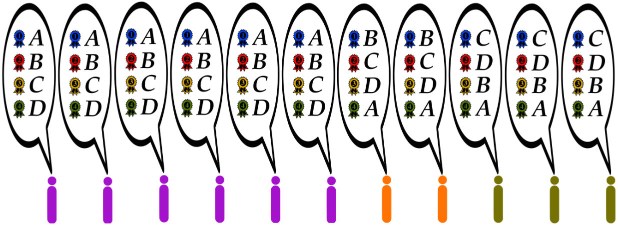

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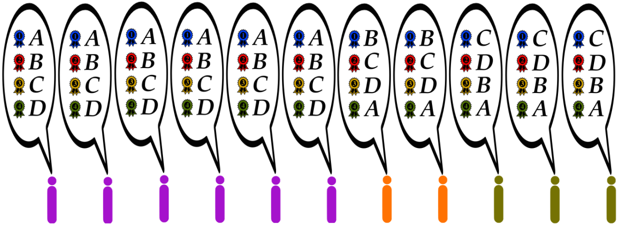
3rd choice	C: $2 \times 6 = 12$	D: $2 \times 2 = 4$	B: $2 \times 3 = 6$
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

A gets $24 + 2 + 3 = 29$ points

The Borda Count Method: Sample election



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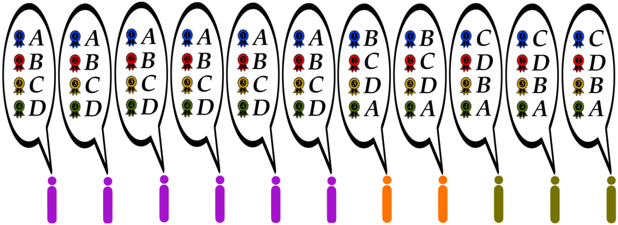
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

A gets $24 + 2 + 3 = 29$ points,
B gets $18 + 8 + 6 = 32$ points

The Borda Count Method: Sample election



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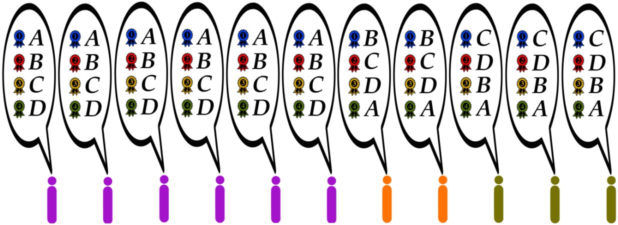
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

A gets $24 + 2 + 3 = 29$ points,
B gets $18 + 8 + 6 = 32$ points,
C gets $12 + 6 + 12 = 30$ points

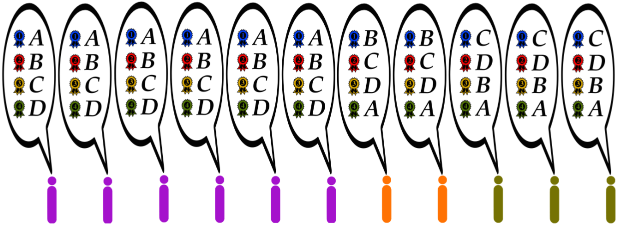
The Borda Count Method: Sample election



	6	2	3
1st choice	A: $4 \times 6 = 24$	B: $4 \times 2 = 8$	C: $4 \times 3 = 12$
2nd choice	B: $3 \times 6 = 18$	C: $3 \times 2 = 6$	D: $3 \times 3 = 9$
3rd choice	C: $2 \times 6 = 12$	D: $2 \times 2 = 4$	B: $2 \times 3 = 6$
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

A gets $24 + 2 + 3 = 29$ points,
B gets $18 + 8 + 6 = 32$ points,
C gets $12 + 6 + 12 = 30$ points,
D gets $6 + 4 + 9 = 19$ points.

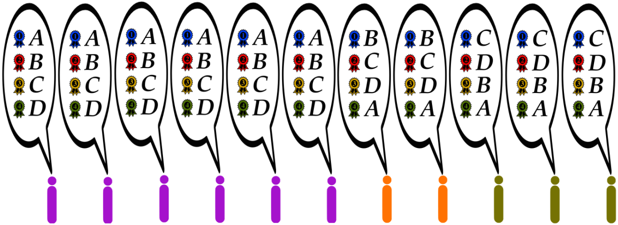
And the winner is...



	6	2	3
1st choice	A: $4 \times 6 = 24$	B: $4 \times 2 = 8$	C: $4 \times 3 = 12$
2nd choice	B: $3 \times 6 = 18$	C: $3 \times 2 = 6$	D: $3 \times 3 = 9$
3rd choice	C: $2 \times 6 = 12$	D: $2 \times 2 = 4$	B: $2 \times 3 = 6$
4th choice	D: $1 \times 6 = 6$	A: $1 \times 2 = 2$	A: $1 \times 3 = 3$

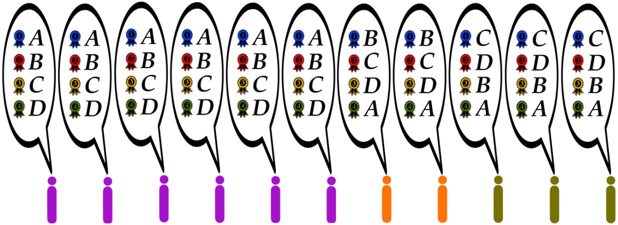
A gets $24 + 2 + 3 = 29$ points,
B gets $18 + 8 + 6 = 32$ points,
C gets $12 + 6 + 12 = 30$ points,
D gets $6 + 4 + 9 = 19$ points.

The Borda count and the Majority Criterion



- Is there a majority candidate?
- Does the Borda count method violate the Majority Criterion?

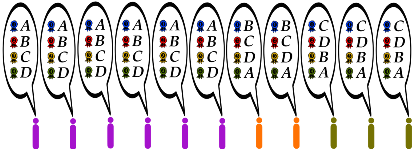
The Borda count and the Majority Criterion



- Is there a majority candidate?
- Does the Borda count method violate the Majority Criterion?

Is there a Condorcet candidate?

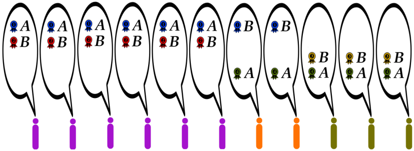
A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.



number of voters:	6	2	3
1st choice	*A*	*B*	*C*
2nd choice	*B*	*C*	*D*
3rd choice	*C*	*D*	*B*
4th choice	*D*	*A*	*A*

Is there a Condorcet candidate?

A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.

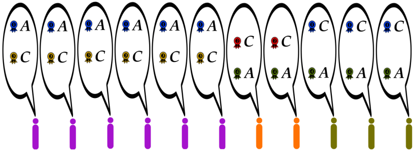


number of voters:	6	2	3
1st choice	*A*	*B*	
2nd choice	*B*		
3rd choice			*B*
4th choice		*A*	*A*

In a two-way race, 6 voters would prefer *A* over *B*, while only 5 would prefer *B* over *A*.

Is there a Condorcet candidate?

A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.

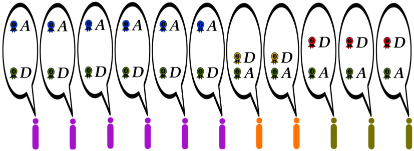


number of voters:	6	2	3
1st choice	A		C
2nd choice		C	
3rd choice	C		
4th choice		A	A

In a two-way race, 6 voters would prefer A over C, while only 5 would prefer C over A.

Is there a Condorcet candidate?

A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.

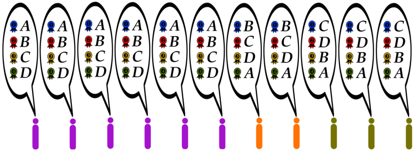


number of voters:	6	2	3
1st choice	*A*		
2nd choice			*D*
3rd choice		*D*	
4th choice	*D*	*A*	*A*

In a two-way race, 6 voters would prefer *A* over *D*, while only 5 would prefer *D* over *A*.

Is there a Condorcet candidate?

A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.



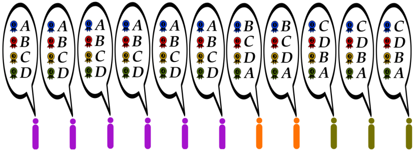
number of voters:	6	2	3
1st choice	*A*	*B*	*C*
2nd choice	*B*	*C*	*D*
3rd choice	*C*	*D*	*B*
4th choice	*D*	*A*	*A*

A is a Condorcet candidate.

A would win in any two-way race against one of the other candidates.

Is there a Condorcet candidate?

A **Condorcet candidate** is a candidate that is favored over every other candidate in a head-to-head matchup.



number of voters:	6	2	3
1st choice	*A*	*B*	*C*
2nd choice	*B*	*C*	*D*
3rd choice	*C*	*D*	*B*
4th choice	*D*	*A*	*A*

A is both a Condorcet candidate
and a majority candidate.

But *B* won the election...

Condorcet Criterion

Fairness Criterion #2: The Condorcet Criterion

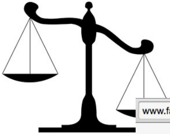
A candidate that beats each of the other candidates in a head-to-head matchup should be the winner.

Condorcet Criterion

The Borda count is not a fair voting scheme

The Borda count method violates both the Majority Criterion and the Condorcet Criterion.

Condorcet Criterion



The
The
Maj

www.fairsportsrules.com/fair-sports-rules-blog/should-baseball-change-its-mvp-voting-system

Should Baseball Change its MVP Voting System?

1/7/2014 0 Comments

There's something wrong with a voting system where a majority of voters can vote for a candidate, and he can still lose.

By Devin McCarthy and Matt Dewilde

Cabrera vs. Trout / ESPN

Major League Baseball decides its MVP using a ranked-choice system, of sorts. It is a point-based system known as a "Borda count" – a method already discussed on this blog in the context of FIFA Ballon D'Or voting. It's a pretty good voting system for baseball, as it allows voters to fully express their MVP preferences without any fear of vote splitting or wasting a vote on a "spoiler" player.

Condorcet Criterion

The plurality method is not a fair voting scheme

The plurality method violates the Condorcet Criterion.

The plurality method violates the Condorcet criterion

Ex. The marching band at Tasmania State University has been invited to perform at five different bowl games: the *R*ose Bowl, the *H*ula Bowl, the *F*iesta Bowl, the *S*ugar Bowl, and the *O*range Bowl. The following preference schedule shows the results of an election held among the 100 members of the band, to be decided by the plurality method.

number of voters:	49	48	3
1st choice	*R*	*H*	*F*
2nd choice	*H*	*S*	*H*
3rd choice	*F*	*O*	*S*
4th choice	*O*	*F*	*O*
5th choice	*S*	*R*	*R*

- Is there a Condorcet candidate?
- Who wins the election?

The Method of Pairwise Comparisons

Perhaps it would be best to decide elections on the basis of head-to-head matchups (or **pairwise comparisons**).

To find the winner of an election under the **method of pairwise comparisons**:

- List all the possible pairwise comparisons.
- For each pairwise comparison, give 1 point to the winner, and 0 points to the loser.
- If there is a tie, give each candidate $\frac{1}{2}$ point.

The Method of Pairwise Comparisons: Sample election

Ex. Five athletes compete for an award that is decided by a poll of sportswriters. The winner will be chosen by the method of pairwise comparisons.

1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A
number of voters:	2	6	4	1	1	4	4

The Method of Pairwise Comparisons: Sample election

$A(7)$; $B(15)$

1st choice	A	B	B				
2nd choice		A	A	B		A	
3rd choice				A	A		
4th choice	B				B		B
5th choice						B	A
number of voters:	2	6	4	1	1	4	4

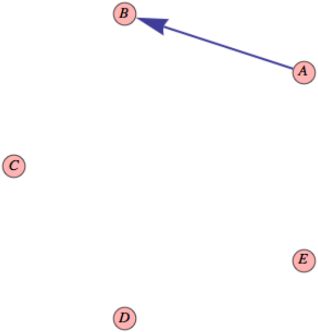
A:

B: |

C:

D:

E:



The Method of Pairwise Comparisons: Sample election

$A\,(16)$; $C\,(6)$

1st choice	A			C	C		
2nd choice		A	A			A	C
3rd choice	C	C		A	A		
4th choice						C	
5th choice			C				A
number of voters:	2	6	4	1	1	4	4

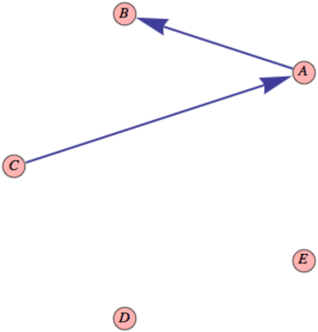
A: |

B: |

C:

D:

E:



The Method of Pairwise Comparisons: Sample election

$A\,(13)$; $D\,(9)$

1st choice	A					D	
2nd choice	D	A	A		D	A	
3rd choice			D	A	A		D
4th choice		D		D			
5th choice							A
number of voters:	2	6	4	1	1	4	4

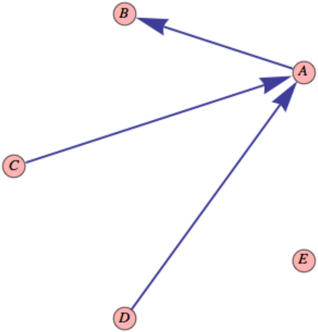
A: ||

B: |

C:

D:

E:



The Method of Pairwise Comparisons: Sample election

$A\,(18);\ \ E\,(4)$

1st choice	A						E
2nd choice		A	A			A	
3rd choice				A	A	E	
4th choice			E				
5th choice	E	E		E	E		A
number of voters:	2	6	4	1	1	4	4

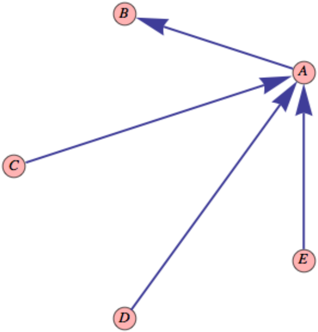
A: ||||

B: |

C:

D:

E:



The Method of Pairwise Comparisons: Sample election

$B\,(10);\;\; C\,(12)$

1st choice		B	B	C	C		
2nd choice				B			C
3rd choice	C	C					
4th choice	B				B	C	B
5th choice			C			B	
number of voters:	2	6	4	1	1	4	4

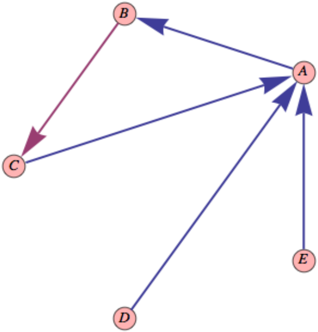
A: ||||

B: |

C: |

D:

E:



The Method of Pairwise Comparisons: Sample election

$B\,(11);\;\; D\,(11)$

1st choice		B	B			D	
2nd choice	D			B	D		
3rd choice			D				D
4th choice	B	D		D	B		B
5th choice						B	
number of voters:	2	6	4	1	1	4	4

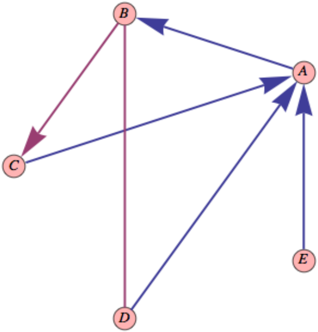
A: ||||

B: | $\frac{1}{2}$

C: |

D: $\frac{1}{2}$

E:



The Method of Pairwise Comparisons: Sample election

$B\,(14);\;\; E\,(8)$

1st choice		B	B				E
2nd choice				B			
3rd choice						E	
4th choice	B		E		B		B
5th choice	E	E		E	E	B	
number of voters:	2	6	4	1	1	4	4

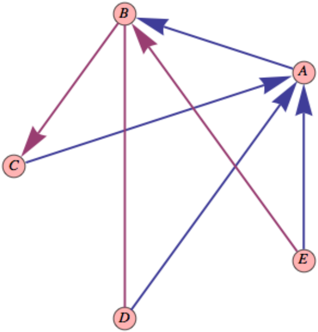
A: ||||

B: $||\frac{1}{2}$

C: |

D: $\frac{1}{2}$

E:



The Method of Pairwise Comparisons: Sample election

$C\,(12);\;\; D\,(10)$

1st choice				C	C	D	
2nd choice	D				D		C
3rd choice	C	C	D				D
4th choice		D		D		C	
5th choice			C				
number of voters:	2	6	4	1	1	4	4

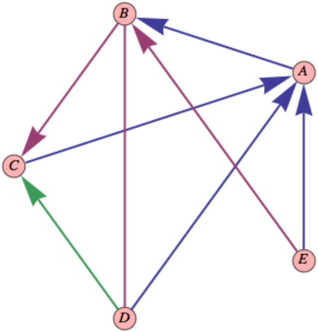
A: |||

B: || $\frac{1}{2}$

C: ||

D: $\frac{1}{2}$

E:



The Method of Pairwise Comparisons: Sample election

$$C\,(10);\;\; E\,(12)$$

1st choice				C	C		E
2nd choice							C
3rd choice	C	C				E	
4th choice			E			C	
5th choice	E	E	C	E	E		
number of voters:	2	6	4	1	1	4	4

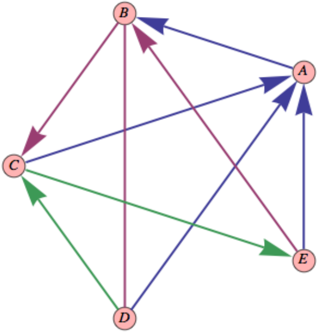
A: $||||$

B: $||\frac{1}{2}$

C: $||$

D: $\frac{1}{2}$

E: $|$



The Method of Pairwise Comparisons: Sample election

$D\,(18)$; $E\,(4)$

1st choice						D	E
2nd choice	D				D		
3rd choice			D			E	D
4th choice		D	E	D			
5th choice	E	E		E	E		
number of voters:	2	6	4	1	1	4	4

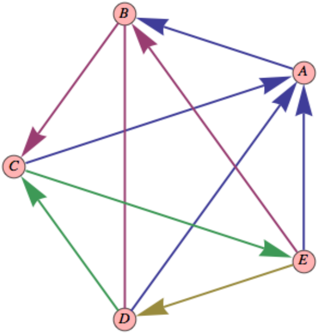
A: $||||$

B: $||\frac{1}{2}$

C: $||$

D: $|\frac{1}{2}$

E: $|$



The Method of Pairwise Comparisons: Sample election

1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A
number of voters:	2	6	4	1	1	4	4

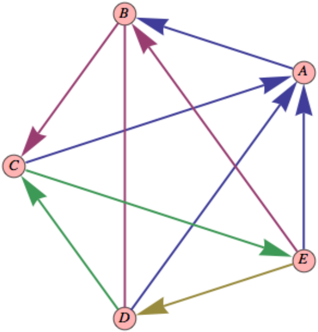
A: $||||$

B: $||\frac{1}{2}$

C: $||$

D: $|\frac{1}{2}$

E: $|$



The Method of Pairwise Comparisons: Sample election

1st choice	A	B	B	$\cancel{C}$	$\cancel{C}$	D	E
2nd choice	D	A	A	B	D	A	$\cancel{C}$
3rd choice	$\cancel{C}$	$\cancel{C}$	D	A	A	E	D
4th choice	B	D	E	D	B	$\cancel{C}$	B
5th choice	E	E	$\cancel{C}$	E	E	B	A
number of voters:	2	6	4	1	1	4	4

Unfortunately, one of the candidates turns out to be ineligible...

The Method of Pairwise Comparisons: Sample election

1st choice	A	B	B			D	E
2nd choice	D	A	A	B	D	A	
3rd choice			D	A	A	E	D
4th choice	B	D	E	D	B		B
5th choice	E	E		E	E	B	A
number of voters:	2	6	4	1	1	4	4

Unfortunately, one of the candidates turns out to be ineligible.

We delete the candidate from the preference schedule,

The Method of Pairwise Comparisons: Sample election

Arrow's Impossibility Theorem

Julie Carmela La Corte

Introduction
Arrow's Impossibility Theorem
Preference Schedules

Voting schemes
Plurality Method
Borda Count Method
The Method of Pairwise Comparisons
Instant runoff voting

Group work

1st choice	A	B	B	B	D	D	E
2nd choice	D	A	A	A	A	A	D
3rd choice	B	D	D	D	B	E	B
4th choice	E	E	E	E	E	B	A
number of voters:	2	6	4	1	1	4	4

Unfortunately, one of the candidates turns out to be ineligible.

We delete the candidate from the preference schedule, and obtain a new preference schedule.

Will this affect the outcome?

The Method of Pairwise Comparisons: Sample election

1st choice	A	B	B	B	D	D	E
2nd choice	D	A	A	A	A	A	D
3rd choice	B	D	D	D	B	E	B
4th choice	E	E	E	E	E	B	A
number of voters:	2	6	4	1	1	4	4

Unfortunately, one of the candidates turns out to be ineligible.

We delete the candidate from the preference schedule, and obtain a new preference schedule.

Will this affect the outcome?

The Method of Pairwise Comparisons: Sample election

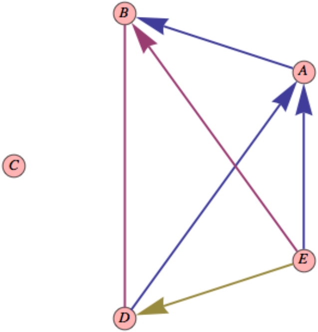
1st choice	A	B	B	B	D	D	E
2nd choice	D	A	A	A	A	A	D
3rd choice	B	D	D	D	B	E	B
4th choice	E	E	E	E	E	B	A
number of voters:	2	6	4	1	1	4	4

A: $||$

B: $||\frac{1}{2}$

D: $|\frac{1}{2}$

E: 0



Independence-of-Irrelevant-Alternatives Criterion

Fairness Criterion #3: The Independence-of-Irrelevant-Alternatives Criterion

The winner of an election should not be hurt by one of the losers dropping out.

Instant runoff voting

- Some municipalities require that a candidate obtain a majority of the first-place votes to be elected. When there are three or more candidates, quite often there is no majority candidate.
- A **run-off election** is typically held at this point: the last place candidate is eliminated from the ballot, and a new election is held.
- The method of **instant runoff voting** (a.k.a. **plurality-with-elimination**) is a more efficient way to implement the same process. This method has become somewhat of a trend in recent years.
- Voters fill out a preference ballot so that they do not need to vote over and over. From the original preference schedule, we eliminate the candidates with the fewest first-place votes one at a time until one of them gets a majority.

Instant runoff voting

- **Round 1.**
  Count the first-place votes for each candidate. If a candidate has a majority of first-place votes, then that candidate is the winner. Otherwise, eliminate the candidate (or candidates if there is a tie) with the fewest last-place votes.

- **Round 2.**
  Cross out the names of any candidates eliminated from the preference schedule, and recount the first-place votes. If a candidate has a majority of first-place votes, then that candidate is the winner. Otherwise, eliminate the candidate (or candidates if there is a tie) with the fewest last-place votes.

- **Round 3.**
  Repeat Round 2 until a winner is found.

Instant runoff voting

Ex. The cities of *A*thens, *B*arcelona, and *C*algary are competing to be the host city for the 2020 Olympics.

A secret vote of the 29 members of the Executive Council of the International Olympic Committee is to be held.

Two days before the actual election, a *straw poll*[1] is held.

Preference schedule for straw poll

1st choice	*A*	*B*	*C*	*A*
2nd choice	*B*	*C*	*A*	*C*
3rd choice	*C*	*A*	*B*	*B*
number of voters:	7	8	10	4

(It turns out that Calgary wins this straw poll.)

[1] A **straw poll** is an unofficial vote or poll indicating the trend of opinion on a candidate or issue.

Instant runoff voting: Sample election

When word gets out that Calgary is favored to win the election, the four delegates represented by the rightmost column of the straw poll's preference schedule decide to switch their votes and vote for Calgary first.

Preference schedule for actual election

1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B
number of voters:	7	8	14

Now who wins?

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B
number of voters:	7	8	14

- *Majority*: More than $(7 + 8 + 14) \div 2$

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B
number of voters:	7	8	14

- *Majority*: More than $(7 + 8 + 14) \div 2 = 14.5$

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	~~A~~	B	C
2nd choice	B	C	~~A~~
3rd choice	C	~~A~~	B
number of voters:	7	8	14

- *Majority*: More than 14.5

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	B	B	C
2nd choice	C	C	B
number of voters:	7	8	14

- *Majority*: More than 14.5

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	B	B	C
2nd choice	C	C	B
number of voters:	7	8	14

- *Majority*: More than 14.5
- B: 15
- C: 14

Instant runoff voting: Sample election

Preference schedule for actual election

1st choice	B	B	C
2nd choice	C	C	B
number of voters:	7	8	14

- *Majority*: More than 14.5
- B: 15 
- C: 14 

Monotonicity Criterion

Fairness Criterion #4: The Monotonicity Criterion

A voter should not be able to hurt a candidate by moving her up in his ballot.

Summary

Violations of the Fairness Criteria

Fairness Criterion	Plurality	Borda count	Pairwise comp.	Instant runoff
Majority	—	✓	—	—
Condorcet	✓	✓	—	✓
Indep.-of-Irrel.	✓	✓	✓	✓
Monotonicity	—	—	—	✓

A checkmark (✓) indicates that the voting scheme violates the Criterion.