MATH 1001 Worksheet #5 | Percentages and Capital

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Contents

A. Solving Percentage Problems.

(A1) Setting up Equations for Percentage Problems

Recall:

A **proportion equation**[a] is an equation of the form "*fraction equals fraction*": $\dfrac{\square}{\square} = \dfrac{\square}{\square}$.

The word "**percent**" means, "*per hundred*." That is, $5\% = \dfrac{5}{100}$.

- Here we used the fact that the word "*per*" is expressed in arithmetic by division: for example, *miles per hour* $= \dfrac{\text{mi.}}{\text{hr.}}$

Since "*percent*" means "*per hundred*," a statement of the form

"\$10 is 5% of \$200"

can be written as a proportion equation:

$$\frac{5}{100} = \frac{\$10}{\$200} \tag{$\dagger$}$$

We can express in English what Equation (†) says in several different ways:

- The PROPORTION of \$10 to \$200 is $5/100$.
- The RATIO of \$10 to \$200 is $1/20$ $(= 5/100)$.
- The RATE is 5%.

☞ Whenever you see the word "RATE" in a financial context, think "*percentage*."

In general, the statement

"A is $P\%$ of B"

can be translated into symbols as follows:

$$P\% = \frac{P}{100} = \frac{A}{B} \tag{$\star$}$$

Equation ($\star$) can be used to find the value of any one of the three variables P, A, and B, if the other two variables' values are given.

[a] Worksheet 1 defines and discusses "*proportion equations*" and their "*ratios*."

If we solve for each of the three variables, we get three different forms of Equation ($\star$):

$$P\% = \frac{A}{B} \qquad\qquad A = P\% \times B \qquad\qquad B = \frac{A}{P\%} \tag{$\star\star\star$}$$

But memorizing these three different forms isn't necessary. In fact, trying to memorize all three—and when to use which one—can be frustrating and counterproductive.

Many find it easier to learn how to use a single equation, the P–A–B formula *(in box below)*, to solve ALL TYPES OF PERCENTAGE PROBLEMS.

is…

$$\boxed{P\% = \frac{A}{B}}$$

…of

- We call $P\%$ the **percentage**,
- we call A the **amount**, and
- we call B the **base**.

Exercise 1.

(a) Solve[b] the P–A–B formula, $P\% = \frac{A}{B}$, for A.

(b) Rewrite the equation you got in part **(a)**, this time replacing $P\%$ with the fraction $\frac{P}{100}$:

Exercise 2. Fill in the blanks with a word or phrase to show how each of the following statements can be translated into an equation. THEN LABEL EACH BOX as A, B, P, or $P\%$.

- The **sales tax** in Georgia is 4% of the purchase.

$$\boxed{} = \frac{4}{100} \times \boxed{}$$

- The **commission** paid to a salesperson is $P\%$ of how much they sell (in dollars' worth).

$$\boxed{} = \frac{\boxed{}}{100} \times \boxed{}$$

- A **discount** is a percentage of the list price.[c]

$$\boxed{} = \boxed{} \times \boxed{}$$

[b] To **solve an equation for a variable**—let's say, **to solve for** A—means to rewrite the equation with A by itself on one side.

[c] A commodity's **list price** (or **sticker price**) is the price it sells for when it's not on sale, before taxes.

We see that each of the three equations in the previous exercise had the form:

$$A = P\% \times B \tag{$\star\star$}$$

—or equivalently,

$$A = \frac{P}{100} \times B. \tag{$\star\star$}$$

Neither of the two equations marked ($\star\star$) need to be memorized if you know the **P–A–B formula**,

is…

$$\boxed{P\% = \frac{A}{B}}$$

…of

since we can just multiply both sides of the P–A–B formula by B to get Equation ($\star\star$).

Similarly, we can solve the P–A–B formula for B to get $B = \dfrac{A}{P\%}$.

☞ The real question when working percentage problems is not which equation to use. (The P–A–B formula always works.)
The *real* question is, WHAT'S A, AND WHAT'S B?

In a percentage problem, we can determine which quantity goes with which variable using the following phrase:

The **amount** $\boldsymbol{A}$ is $P\%$ of the **base** $\boldsymbol{B}$.

Exercise 3. Aleah is driving for Lyft. One day, she puts $\$20$ of gas in her tank, and drives until it's all used up. Lyft pays her $\$75$ for the day's work. That leaves her $\$75 - \$20 = \$55$ ahead of where she started.

(a) Use the P–A–B formula to set up a fraction that expresses the $\$55$ gain as a percentage of how much Aleah spent in gas.

(Hint: What are we taking the percent of? That's the base, B.)

(b) Find $P\% = P/100$ as a decimal.

(c) If Aleah buys $\$45$ of gas, and drives for Lyft until it's all used up, how much (*in dollars*) can she expect to end up ahead?

Exercise 4. *(Optional.)* The fraction

$$P\% = \frac{\text{amount Aleah ends up ahead}}{\text{amount Aleah spent on gas}}$$

is Aleah's **rate of return**. Write Aleah's rate of return as a decimal, and explain in plain English how Aleah can use this number to predict how much she ends up ahead, for any given amount spent on gas.

Exercise 5. The AMOUNT A of *sales tax*, *commission*, and *discount* can all be figured out using the P–A–B formula:

$$P\% = \frac{A}{B}$$

$$A = P\% \times B \qquad \text{(solve for } A\text{)}$$

$$A = \frac{P}{100} \times B \qquad \text{(convert } P\% \text{ to a decimal for the calculator)}$$

In percentage problems, we must remember what is to be *done* with the amount:

amount of...	**what we do with it**	
Discount	is SUBTRACTED from the list price	to get the total cost to the buyer.
Sales tax	is ADDED to the sale price	to get the total cost to the buyer.
Commission	is PAID to the salesperson.	

(a) Ann Taylor is selling a lovely pair of pants with a list price of $\$109$.

- What's the AMOUNT of the discount if the pants are on sale for 40% off?
- How much SALES TAX is charged on the discounted price? (Use a 4% rate of sales tax.)
- How much will the pants cost the customer, after the discount and sales tax are applied?

(b) Suppose a real estate agent in Atlanta gets paid a commission of 5.87% on every house she sells.

- How much commission will she be paid for selling a $\$170,000$ house[d]?
- How much commission will she be paid for selling a $\$675,000$ house[e]?

[d] Price retrieved Jun. 19, 2021 from Zillow.com: 3 bedrooms, 1 bath, single-family house (1,363 sq. ft.), lot: 5,293 sq. ft.
[e] Price retrieved Jun. 19, 2021 from Zillow.com: 3 bedrooms, 2.5 baths, single-family house (2,781 sq. ft.), lot: 8,712 sq. ft.

B. Future Value.

(B1) Initial Advance (Principal) and Future Value

For our purposes in this class,[1] the word "**capital**" means,[2] money invested or loaned for the purpose of making a profit: "money that breeds more money."

For example:

- Suppose a landlord wants to invest $\$2,000,000$ in real estate.
- He buys a multi-unit apartment building with the $\$2,000,000$, and collects

  $$\$192,000 \text{ in rent annually}$$

  from the tenants.[f]

The **initial advance—**also known as the **principal**, P—is the amount initially invested. In this case,

$$P = \$2,000,000,$$

the amount advanced to buy the property.[3]

☞ The PROFIT in this case is the annual rent collected.[g]

The investment's **future value** after t years is the principal, plus the total profit accumulated over the course of t years.

Exercise 6. Let's assume the amount of rent collected by the landlord is held *constant*—that is, he never raises or lowers the rent.

(a) What's the landlord's total profit after 2 years?

(b) What's the landlord's total profit after t years?

(c) Write an algebraic model ($A = \cdots$) for the FUTURE VALUE of the property, A, using P to stand for the principal, and using t for the number of years.

[f] 16 households $\times$ $\$1000$ rent per household per month $\times$ 12 months per year, if you're wondering where that number came from.
[g] For simplicity's sake, we've assumed the landlord has zero expenses (does no repairs, pays no taxes, never gets sued, etc.).

(d) Use your model to find the future value after 6 months. (Careful! We're measuring time, t, in years, so we'll have to convert 6 months to $t = ?$ years.)

(e) What's the shape of the graph of your algebraic model?

(f) Sketch the graph of your algebraic model, with future value A on the vertical axis (up & down), and time t on the horizontal axis (left & right).

(g) Assuming the building will sell for exactly what the landlord paid for it, we can use our model to predict how long it will take for the future value, A, to grow to any amount we like. How many years should the landlord wait to sell the building if he wants to *double* his initial investment?

(B2) Principal, Interest, and Future Value

Whether or not a loan counts as "capital"—money that breeds money—depends on whether you're collecting interest, or paying it.

- From the BANKER'S perspective, *a loan is an investment*, made for the purpose of collecting a PROFIT.[4]

Recall that the amount initially advanced is called the *principal*, P.

The **future value of a loan or investment** is the total amount, A, paid back to the investor or lender.

- If $A > P$, a profit was collected.
- If $A = P$, the profit was 0.
- If $A < P$, the loan or investment was "bad"—it ended up paying back less money than was initially advanced.

The **interest**, I, is the total paid back, minus the principal initially advanced.

In other words, *the interest is the profit*—that is, the amount of "return" on the loan or investment:

$$\underset{\substack{\text{future}\\\text{value}}}{A} = \underset{\substack{\text{principal}\\\text{(initial}\\\text{advance)}}}{P} + \underset{\substack{\text{interest}\\\text{(profit)}}}{I} \qquad (\ddagger)$$

From the BORROWER'S perspective:

- *A loan is rented money.*
- They advance you some cash initially (P)...
- ...but you'll pay back more in the long run (a total of $P + I$, the principal plus the interest).

Formula (‡) applies not just to loans, but also to other types of investment, including pensions, annuities, and savings accounts.

Exercise 7. A pawn shop loans Julie $\$75$ and holds her Fender Stratocaster guitar as collateral.[h] The interest on the loan is $\$15$ per month.

(a) What's P, in this situation?

(b) How much interest must be paid over the course of one year?

(c) What's the future value (after 1 year) of the loan to the pawn shop?

(d) What PERCENTAGE of the future value (after 1 year) is the interest?

[h] The **collateral** for a loan is something of value that the lender owns, which can legally be seized if the lender does not pay on time.

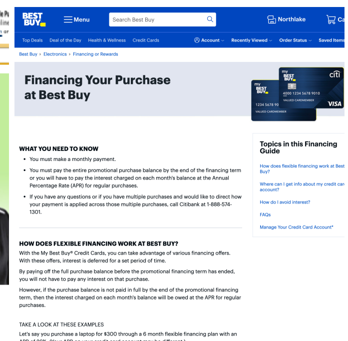
Once upon a time, large household appliances (like refrigerators and stoves) could be financed using a **simple interest loan** at many department stores.

- The amount of **simple interest** accumulated over the course of t years is given by the formula

$$I = Prt$$

where P is the principal (initial advance), and r is the annual rate of interest.

☞ The annual rate of interest on a loan or investment is often called the **APR**, which stands for **annual percentage rate**.



Nowadays, simple interest loans are most often encountered:

- at pawn shops,
- as specialized "luxury" credit accounts, and
- in math textbooks.

Exercise 7. When he was a younger man, your grandfather borrowed \$200 to finance a videocassette recorder at an annual rate of simple interest of 8.25% for one and a half years.

- How much interest did he pay?
- How much did he repay altogether?

Exercise 8. Suppose that, over the course of 5 months, Julie pays $\$75$ in interest on a pawn shop loan for an initial advance of $\$100$. Find the APR (annual percentage rate, r) using the following steps:

- Convert "5 *months*" to YEARS to get t.
- Use the **simple interest formula**, $I = Prt$, to find r. (You may find it helpful to solve for r first—that is, to use algebra to rewrite the equation with r by itself on one side, *before* "plugging in" numbers.)
- The simple interest formula gives r as a *decimal*. But in the real world, the APR, r, is always given as a *percentage*. What's r, as a *percentage*?
- How much *in interest* is Julie expected to repay over the course of a full year? (Now that we know r, the simple interest formula can be used to answer this question.)

The equation

$$A = P + I$$

can be used for *any* type of capital ("money, P, that grows into a larger amount, $P + I$, by being invested or loaned").

But the equation

$$I = Prt$$

can only be used for SIMPLE INTEREST (and *not for*, for instance, credit card debt, mortgages, student loans, or annuities).

Combining these two equations gives us the **future value formula for simple interest**:

$$\boxed{A = P \cdot (1 + rt)}$$

Exercise 9. You are a loan officer at a bank. A customer requests a 4-year loan of $\$9,000$ to pay for their child's medical care. You offer[i] the customer an APR of 11.3%.

(a) From the bank's perspective, what's the future value of the loan after four years?

(b) How much of that is interest?

(c) What PERCENTAGE of the future value (after 4 years) is the interest?

Exercise 11. *(Optional)* Use algebra to show that the future value formula for simple interest, $A = P \cdot (1 + rt)$, can be obtained by combining the general future value formula, $A = P + I$, with the simple interest formula, $I = Prt$.

[i] As of Jun. 19, 2021, Georgia bank Delta Community Credit Union advertises personal loans for up to 60 months at an APR between 5.50% and 16.75%.

Endnotes

[1] Technically, "*capital*" can refer either to *money* or to *what money has bought*—for example, a $\$2,000,000$ apartment building purchased by a landlord.

Since property enters our equations only in the form of its cash value, we'll ignore the distinction between $\$2,000,000$ of capital "in money form" and $\$2,000,000$ of capital in the form of *purchased stuff*.

[2] Eighteenth-century economist Adam Smith defined "*capital*" as follows:

> "[A man's] whole stock [...] is distinguished into two parts. That part which, he expects, is to afford him [...] revenue, is called his **capital**. The other is that which supplies his immediate consumption [...]" (Smith, *Wealth of Nations*, 1776).

Another classical economist, Reverend Thomas R. Malthus, later clarified:

> "The capitalist . . . *expects* an equal profit upon all parts of the capital which he advances." (Malthus, *Principles of Political Economy*, 1836, as quoted in Marx, *Capital*, Vol. 3, Ch. 1, Marx's italics)

A contemporary source states that:

> "The capital of a business is the money it has available to pay for its day-to-day operations and to fund its future growth. [...] In a broader sense, the term may be expanded to include all of a company's assets that have monetary value, such as its equipment, real estate, and inventory. But when it comes to budgeting, capital is cash flow. [...] *Capital is used by companies to pay for the ongoing production of goods and services in order to create profit.* [...] Labor and building expansions are two common areas of capital allocation. *By investing capital, a business or individual seeks to earn a higher return than the capital's costs.*" (M. Hargrave, *Investopedia.com*, "Capital," updated Mar. 9, 2021; italics added)

[3] "'**Property**' describes one's exclusive right to possess, use, and dispose of a thing [...]" (S. H. Gifis, *Law Dictionary*, Barron's Educational Series, 1996)

[4] When money is loaned to a BUSINESS, the idea is that the lender takes a share in the business's profits—the lender's share is the *interest*.

- "The rate of interest depends (1) on the rate of profit; (2) on the proportion in which the entire profit is divided between the lender and borrower" (*The Economist*, Jan. 22, 1853, as quoted in Marx, *Capital*, Vol. 3, Ch. 22)
- "That a man who borrows money with a view of making a profit by it, should give some portion of this profit to the lender, is a self-evident principle of natural justice" (J. Gilbart, *The History and Principles of Banking*, 1834, as quoted in Marx, op. cit., Ch. 21)
- "That which men pay as interest for the use of what they borrow, is a part of the profit it is capable of producing." (J. Massie, 1750, as quoted in Marx, op. cit., Ch. 21)
- "Let us take the average rate of profit as 20 per cent. Under average conditions, then, and with the average level of intelligence and activity appropriate to the intended purpose, a machine with a value of £100 that is applied as capital yields a profit of £20. Thus a man who has £100 at his disposal holds in his hands the power of making this £100 into £120, and thus producing a profit of £20. [...] If this man makes over his £100 for a year to someone else, who actually does use it as capital, he gives him the power to produce £20 profit, [...] If the second man pays the proprietor of the £100 a sum of £5, say, at the end of the year, i.e. a portion of the profit produced, [...then t]he part of the profit paid in this way is called **interest**. [...] The £100 produces a profit of £20 by functioning as capital, whether industrial or commercial. [...] buying has [...] been transformed into lending, and price into a share in the profit." (as quoted in Marx, op. cit., Ch. 21)